CHINESE JOURNAL OF PHYSICS VOL. 43, NO. 6 DECEMBER 2005

Anisotropic Cosmological Models with Perfect Fluid and Dark Energy

Bijan Saha*

Laboratory of Information Technologies,
Joint Institute for Nuclear Research, Dubna,
141980 Dubna, Moscow Region, Russia
(Received February 9, 2005)

We consider a self-consistent system of Bianchi type-I (BI) gravitational fields and a binary mixture of a perfect fluid and dark energy. The perfect fluid is taken to be one obeying the usual equation of state, i.e., $p = \zeta\varepsilon$, with $\zeta \in [0, 1]$, whereas the dark energy density is considered to be either quintessence or the Chaplygin gas. Exact solutions to the corresponding Einstein equations are obtained.

PACS numbers: 04.20.Ha, 03.65.Pm, 04.20.Jb

I. INTRODUCTION

The description of the different phases of the Universe concerning the time evolution of its acceleration field is among the main objectives of the cosmological models. There is mounting evidence that the Universe at present is dominated by the so-called dark energy. Although the nature of the dark energy (DE) is currently unknown, it is felt that it is non-baryonic in origin [1]. It is also believed that the dark energy has a large negative pressure that leads to an accelerated expansion of the Universe.

In view of its importance in explaining observational cosmology, many authors have considered cosmological models with dark energy. The simplest example of dark energy is the cosmological constant, introduced by Einstein in 1917 [2]. The discovery that the expansion of the Universe is accelerating [3] has promoted the search for new types of matter that can behave like a cosmological constant [4, 5], by combining a positive energy density and negative pressure. This type of matter is often called *quintessence*. Zlatev *et al.* [6] showed that the "tracker field", a form of quintessence, may explain the coincidence, adding new support to the quintessence scenario.

An alternative model for the dark energy density was used by Kamenshchik *et al.* [7], where the authors suggested the use of some perfect fluid that obeys an "exotic" equation of state. This type of matter is known as a *Chaplygin gas*. The fate of density perturbations in a Universe dominated by a Chaplygin gas, which exhibits negative pressure, was studied by Fabris *et al.* [8]. Models with Chaplygin gas were also studied in Refs. [9, 10]. In a recent paper Kremer [11] has modelled the Universe as a binary mixture, with constituents described by a van der Waals fluid and by a dark energy density. In doing so the authors considered mainly a spatially flat, homogeneous and isotropic Universe that can be described by a Friedmann-Robertson-Walker (FRW) metric.

The theoretical arguments and recent experimental data, which support the existence of an anisotropic phase that approaches an isotropic one, leads us to consider models of the Universe with an anisotropic background. The simplest of the anisotropic models, which nevertheless rather completely describes the anisotropic effects, are Bianchi type-I (BI) homogeneous models, whose spatial sections are flat but the expansion or contraction rate is direction-dependent. Moreover, a BI universe falls within the general analysis of the singularity given by Belinskii *et al.* [12] and evolves into a FRW universe [13] in the presence of matter obeying the equation of state $p = \zeta\varepsilon$, $\zeta < 1$. Since the modern-day Universe is almost isotropic on the large scale, this feature of the BI universe makes it a prime candidate for studying the possible effects of anisotropy in the early Universe based on present-day observations. In a number of papers, e.g., [14, 15], we have studied the role of a nonlinear spinor and/or a scalar fields in the formation of an anisotropic Universe free from an initial singularity. It was shown that for a suitable choice of nonlinearity and the sign of the Λ term, the model in question allows for regular solutions and the Universe becomes isotropic during the process of evolution. Recently Khalatnikov *et al.* [16] studied the Einstein equations for a BI Universe in the presence of dust, stiff matter and a cosmological constant. In a recent paper [17] the author studied a self-consistent system with a Bianchi type-I gravitational field with a binary mixture of a perfect fluid and dark energy given by the cosmological constant. The perfect fluid in that paper was chosen to obey either the usual equation of state, i.e., $p = \zeta\varepsilon$, with $\zeta \in [0, 1]$, or a van der Waals equation of state. In this paper we study the evolution of an initially anisotropic Universe given by a BI spacetime and a binary mixture of a perfect fluid obeying the equation of state $p = \zeta\varepsilon$ and dark energy given by either quintessence or a Chaplygin gas.

II. BASIC EQUATIONS

The gravitational field in our case is given by a Bianchi type I metric in the form

$$ds^2 = dt^2 - a^2 dx^2 - b^2 dy^2 - c^2 dz^2 \,, \tag{1}$$

with the metric functions a, b, c being functions of time t only.

We write the Einstein field equations for the BI space-time in the form

$$\frac{\ddot b}{b} + \frac{\ddot c}{c} + \frac{\dot b}{b}\frac{\dot c}{c} = \kappa T_1^1 \,, \tag{2a}$$

$$\frac{\ddot c}{c} + \frac{\ddot a}{a} + \frac{\dot c}{c}\frac{\dot a}{a} = \kappa T_2^2 \,, \tag{2b}$$

$$\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\dot a}{a}\frac{\dot b}{b} = \kappa T_3^3 \,, \tag{2c}$$

$$\frac{\dot a}{a}\frac{\dot b}{b} + \frac{\dot b}{b}\frac{\dot c}{c} + \frac{\dot c}{c}\frac{\dot a}{a} = \kappa T_0^0 \,. \tag{2d}$$

Here κ is the Einstein gravitational constant and the over-dot means differentiation with respect to t. The energy-momentum tensor of the source is given by

$$T_{\mu}^{\nu} = (\varepsilon + p)u_{\mu}u^{\nu} - p\delta_{\mu}^{\nu}\,, \tag{3}$$

where u^{μ} is the flow vector satisfying

$$g_{\mu\nu}u^{\mu}u^{\nu} = 1\,. \tag{4}$$

Here ε is the total energy density of a perfect fluid and/or dark energy density, while p is the corresponding pressure. p and ε are related by an equation of state which will be studied below in detail. In the co-moving system of coordinates, from (3) one finds

$$T_0^0 = \varepsilon, \qquad T_1^1 = T_2^2 = T_3^3 = -p\,. \tag{5}$$

In view of (5), from (2) one immediately obtains [14]

$$a(t) = D_1\tau^{1/3}\exp[X_1\int\frac{dt}{\tau(t)}]\,, \tag{6a}$$

$$b(t) = D_2\tau^{1/3}\exp[X_2\int\frac{dt}{\tau(t)}]\,, \tag{6b}$$

$$c(t) = D_3\tau^{1/3}\exp[X_3\int\frac{dt}{\tau(t)}]\,. \tag{6c}$$

Here D_i and X_i are some arbitrary constants obeying

$$D_1D_2D_3 = 1\,, \qquad X_1 + X_2 + X_3 = 0\,,$$

and τ is a function of t defined to be

$$\tau = abc\,. \tag{7}$$

From (2) for τ one finds

$$\frac{\ddot\tau}{\tau} = \frac{3\kappa}{2}\left(\varepsilon - p\right)\,. \tag{8}$$

On the other hand, the conservation law for the energy-momentum tensor gives

$$\dot\varepsilon = -\frac{\dot\tau}{\tau}\left(\varepsilon + p\right)\,. \tag{9}$$

After a little manipulation, from (8) and (9), we find

$$\dot\tau = \pm\sqrt{C_1 + 3\kappa\varepsilon\tau^2}\,, \tag{10}$$

with C_1 being an integration constant. On the other hand, rewriting (9) in the form

$$\frac{\dot\varepsilon}{(\varepsilon + p)} = -\frac{\dot\tau}{\tau} \tag{11}$$

and taking into account that the pressure and the energy density obey an equation of state of type $p = f(\varepsilon)$, we conclude that ε and p, hence the right hand side of Eq. (8), is a function of τ only, i.e.,

$$\ddot{\tau} = \frac{3\kappa}{2}(\varepsilon - p)\,\tau \equiv \mathcal{F}(\tau)\,. \tag{12}$$

From the mechanical point of view, Eq. (12) can be interpreted as the equation of motion for a single particle with a unit mass under the force $\mathcal{F}(\tau)$. Then the following first integral exists [18]:

$$\dot{\tau} = \sqrt{2[\mathcal{E} - \mathcal{U}(\tau)]}\,. \tag{13}$$

Here $\mathcal{E}$ can be viewed as energy and $\mathcal{U}(\tau)$ is the potential of the force $\mathcal{F}$. Comparing Eqs. (10) and (13) one finds $\mathcal{E} = C_1/2$ and

$$\mathcal{U}(\tau) = -\frac{3}{2}\kappa\varepsilon\tau^2\,. \tag{14}$$

Finally, rearranging (10), we write the solution to Eq. (8) in quadrature form:

$$\int \frac{d\tau}{\sqrt{C_1 + 3\kappa\varepsilon\tau^2}} = t + t_0\,, \tag{15}$$

where the integration constant t_0 can be taken to be zero, since it only gives a shift in time.

In the following we study Eqs. (8) and (9) for a perfect fluid and/or dark energy, for different equations of state obeyed by the source fields.

III. THE UNIVERSE AS A BINARY MIXTURE OF PERFECT FLUID AND DARK ENERGY

In this section we thoroughly study the evolution of the BI Universe filled with a perfect fluid and dark energy. Taking into account that the energy density (ε) and pressure (p) in this case comprise those of a perfect fluid and dark energy, i.e.,

$$\varepsilon = \varepsilon_{\rm pf} + \varepsilon_{\rm DE}\,, \qquad p = p_{\rm pf} + p_{\rm DE}\,,$$

the energy momentum tensor can be decomposed as

$$T_\mu^\nu = (\varepsilon_{\rm DE} + \varepsilon_{\rm pf} + p_{\rm DE} + p_{\rm pf})u_\mu u^\nu - (p_{\rm DE} + p_{\rm pf})\delta_\mu^\nu\,. \tag{16}$$

In the above equation, $\varepsilon_{\rm DE}$ is the dark energy density, $p_{\rm DE}$ is its pressure. We also use the notations $\varepsilon_{\rm pf}$ and $p_{\rm pf}$ to denote the energy density and the pressure of the perfect fluid, respectively. Here we consider the case when the perfect fluid in question obeys the following equation of state

$$p_{\rm pf} = \zeta\,\varepsilon_{\rm pf}\,, \tag{17}$$

where ζ is a constant that lies in the interval $\zeta \in [0, 1]$. Depending on its numerical value, ζ describes the following types of Universes [13] :

$$\zeta = 0\,, \qquad \text{(dust Universe)}\,, \tag{18a}$$

$$\zeta = 1/3\,, \qquad \text{(radiation Universe)}\,, \tag{18b}$$

$$\zeta \in (1/3,\, 1)\,, \qquad \text{(hard Universes)}\,, \tag{18c}$$

$$\zeta = 1\,, \qquad \text{(Zel'dovich Universe or stiff matter)}\,. \tag{18d}$$

In a comoving frame the conservation law of the energy momentum tensor leads to the balance equation for the energy density

$$\dot{\varepsilon}_{\rm DE} + \dot{\varepsilon}_{\rm pf} = -\frac{\dot{\tau}}{\tau}\left(\varepsilon_{\rm DE} + \varepsilon_{\rm pf} + p_{\rm DE} + p_{\rm pf}\right)\,. \tag{19}$$

The dark energy is supposed to interact with itself only, so it is minimally coupled to the gravitational field. As a result, the evolution equation for the energy density decouples from that of the perfect fluid, so from Eq. (19) we obtain two balance equations :

$$\dot{\varepsilon}_{\rm DE} + \frac{\dot{\tau}}{\tau}\left(\varepsilon_{\rm DE} + p_{\rm DE}\right) = 0\,, \tag{20a}$$

$$\dot{\varepsilon}_{\rm pf} + \frac{\dot{\tau}}{\tau}\left(\varepsilon_{\rm pf} + p_{\rm pf}\right) = 0\,. \tag{20b}$$

In view of Eq. (17), from (20b) one easily finds

$$\varepsilon_{\rm pf} = \varepsilon_0/\tau^{(1+\zeta)}\,, \qquad p_{\rm pf} = \varepsilon_0\zeta/\tau^{(1+\zeta)}\,, \tag{21}$$

where ε_0 is the integration constant. In the absence of the dark energy one immediately finds that

$$\tau = Ct^{2/(1+\zeta)}\,, \tag{22}$$

with C being some integration constant. As one can see from (6), in the absence of a Λ term, for $\zeta < 1$, an initially anisotropic Universe eventually evolves into an isotropic FRW one, whereas, for $\zeta = 1$, i.e., in the case of stiff matter, the isotropization does not take place.

In what follows, we consider the case where the Universe is also filled with the dark energy. In Fig. 1 we have plotted the potentials for a Universe filled with a perfect fluid, a perfect fluid plus quintessence and perfect fluid plus Chaplygin gas, respectively. The perfect fluid is given by a radiation. As one can see, these types of potentials allow only infinite motion, i.e., the Universe expands infinitely. Fig. 2 shows the evolution of the BI Universe. The introduction of dark energy results in accelerated expansion of the Universe. The acceleration has been illustrated in Fig. 3.

Fig. 4 shows the evolution of a BI Universe filled with a perfect fluid and a Chaplygin gas. Here “d”, “r”, “h” and “s” stand for dust, radiation, hard Universe, and stiff matter, respectively. As one sees, even in the case of stiff matter the Universe expands rapidy enough to evolve into an isotropic one.

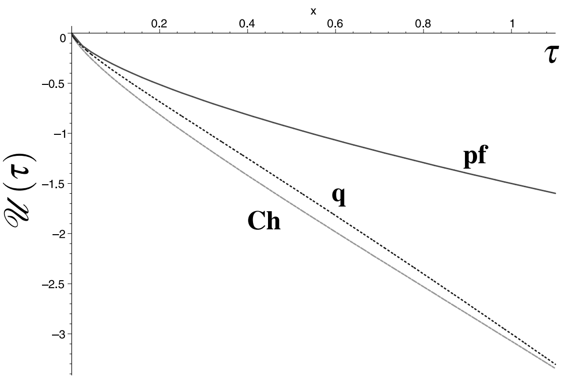

FIG. 1: The potentials when the Universe is filled with a perfect fluid, a perfect fluid plus quintessence and a perfect fluid plus Chaplygin gas, respectively.

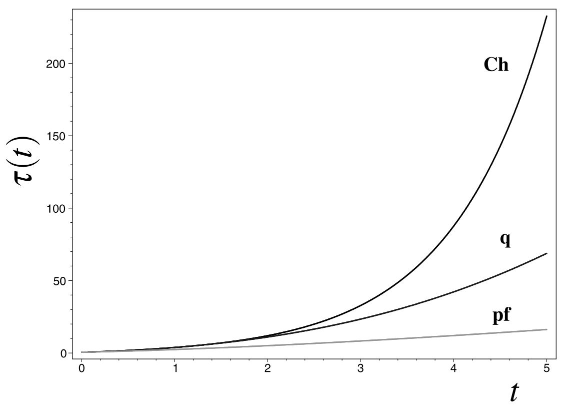

FIG. 2: Evolution of the BI Universe corresponding to the potentials illustrated in Fig. 1.

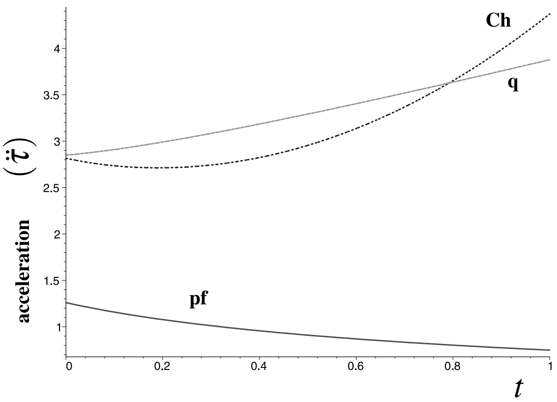

FIG. 3: Acceleration of a BI Universe filled with a perfect fluid, a perfect fluid plus quintessence and a perfect fluid plus Chaplygin gas, respectively.

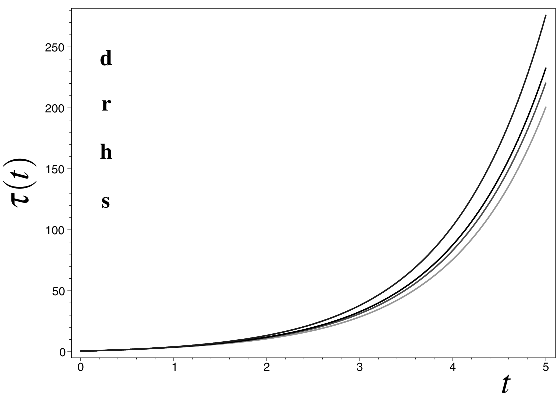

FIG. 4: Evolution of the BI Universe filled with a perfect fluid and Chaplygin gas.

III-1. The case with a quintessence

Let us consider the case when the dark energy is given by a quintessence. As was mentioned earlier, a new type of matter, often known as quintessence, can behave like a cosmological constant. It was constructed by combining a positive energy density and negative pressure and obeys the equation of state

$$p_{\rm q} = w_{\rm q}\varepsilon_{\rm q}\,, \tag{23}$$

where the constant $w_{\rm q}$ varies between -1 and zero, i.e., $w_{\rm q} \in [-1,\, 0]$. On account of (23), from (20a) one finds

$$\varepsilon_{\rm q} = \varepsilon_{0{\rm q}}/\tau^{(1+w_{\rm q})}\,, \qquad p_{\rm q} = w_{\rm q}\varepsilon_{0{\rm q}}/\tau^{(1+w_{\rm q})}\,, \tag{24}$$

with $\varepsilon_{0{\rm q}}$ being some integration constant.

Now the evolution equation for τ (8) can be written as

$$\ddot{\tau} = \frac{3\kappa}{2}\left(\frac{(1-\zeta)\varepsilon_0}{\tau^{\zeta}} + \frac{(1-w_{\rm q})\varepsilon_{0{\rm q}}}{\tau^{w_{\rm q}}}\right)\,. \tag{25}$$

As was mentioned earlier, Eq. (25) admits an exact solution that can be written in quadrature as

$$\int \frac{d\tau}{\sqrt{C_1 + 3\kappa\big(\varepsilon_0\tau^{(1-\zeta)} + \varepsilon_{0{\rm q}}\tau^{(1-w_{\rm q})}\big)}} = t + t_0\,. \tag{26}$$

Here t_0 is a constant of integration that can be taken to be trivial.

III-2. The case with a Chaplygin gas

Let us now consider the case where the dark energy is represented by a Chaplygin gas. We have already mentioned that the Chaplygin gas was suggested as an alternative model of dark energy with an exotic equation of state, namely

$$p_{\rm c} = -A/\varepsilon_{\rm c}\,, \tag{27}$$

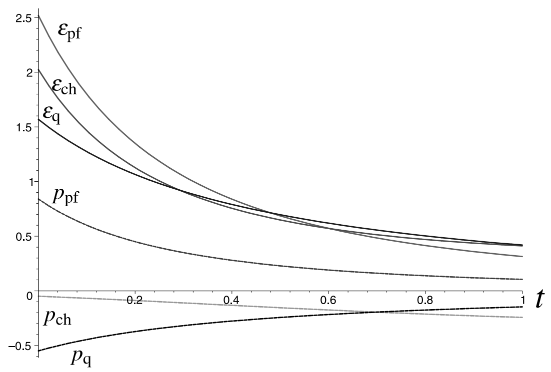

FIG. 5: The energy density and the corresponding pressure when the Universe is filled with a perfect fluid, quintessence and Chaplygin gas, respectively.

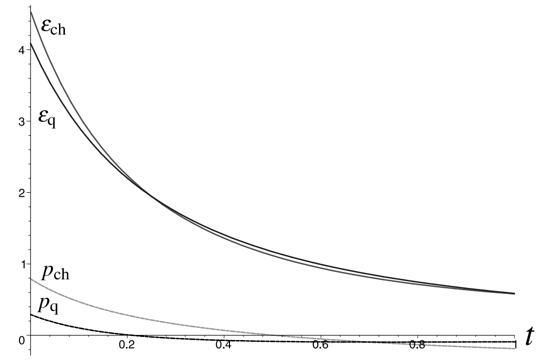

FIG. 6: The energy density and corresponding pressure when the Universe is a binary mixture of a perfect fluid and quintessence and a perfect fluid and Chaplygin gas, respectively.

with A being a positive constant. In view of Eq. (27), from (20a) one now obtains

$$\varepsilon_{\rm c} = \sqrt{\varepsilon_{0{\rm c}}/\tau^2 + A}\,, \qquad p_{\rm c} = -A/\sqrt{\varepsilon_{0{\rm c}}/\tau^2 + A}\,, \tag{28}$$

with $\varepsilon_{0{\rm c}}$ being an integration constant.

Proceeding analogously, as in previous case for τ, we now have

$$\ddot{\tau} = \frac{3\kappa}{2}\left(\frac{(1-\zeta)\varepsilon_0}{\tau^\zeta} + \sqrt{\varepsilon_{0{\rm c}} + A\tau^2} + A/\sqrt{\varepsilon_{0{\rm c}} + A\tau^2}\right)\,. \tag{29}$$

The corresponding solution in quadrature now has the form

$$\int \frac{d\tau}{\sqrt{C_1 + 3\kappa\left(\varepsilon_0\tau^{(1-\zeta)} + \sqrt{\varepsilon_{0{\rm c}}\tau^2 + A\tau^4}\right)}} = t\,, \tag{30}$$

where the second integration constant has been taken to be zero.

IV. CONCLUSION

A self-consistent system of a BI gravitational field filled with a perfect fluid and dark energy has been considered. The exact solutions to the corresponding field equations are obtained. The inclusion of the dark energy into the system gives rise to an accelerated expansion of the model. As a result, the initial anisotropy of the model quickly dies away. Note that the introduction of the dark energy does not eliminate the initial singularity.

References

* Electronic address: `saha@thsun1.jinr.ru,bijan@jinr.ru`
[1] V. Sahni, *Dark Matter and Dark Energy*, e-print astro-ph/0403324.
[2] A. Einstein, Sitzungsber. Preuss. Acad. Wiss. **1**, 142 (1917) and **1**, 349 (1919) [English translation: *The Principle of Relativity* (Methuen, 1923, reprinted by Dover, New York, 1924), pp. 177 and 191].
[3] N. A. Bachall, J. P. Ostriker, S. Perlmutter, and P. J. Steinhardt, Science **284**, 1481 (1999).
[4] R. R. Cladwell, R. Dave, and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998); I. Zlatev, L. Wang, and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).
[5] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000).
[6] I. Zlatev, L. Wang, and P. J. Steinhardt, Phys. Rev. Lett. **85**, 896 (1999).
[7] A. Yu. Kamenshchik, U. Moschella, and V. Pasquier, Phys. Lett. B. **511**, 265 (2001)
[8] J. C. Fabris, S. V. B. Goncalves, and P. E. de Souza, Gen. Rel. Grav. **34**, 53 (2002).
[9] A. Dev, J. S. Alcaniz, and D. Jain, Phys. Rev. D. **67**, 023515 (2003).
[10] M. C. Bento, O. Bertolami, and A. A. Sen, Phys. Rev. D **66**, 043507 (2002)
[11] G. M. Kremer, *Cosmological models described by a mixture of van der Waals fluid and dark energy*, e-print gr-qc/0309111.
[12] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Advances in Physics **19**, 525 (1970).
[13] K. C. Jacobs, Astrophysical Journal **153**, 661 (1968).
[14] Bijan Saha, Phys. Rev. D **64**, 123501 (2001) [arXiv: gr-qc/0107013].
[15] Bijan Saha and Todor Boyadjiev, Phys. Rev. D **69**, 124010 (2004) [arXiv: gr-qc/0311045].
[16] I. M. Khalatnikov and A. Yu. Kamenshchik, Phys. Lett. B **553**, 119 (2003).
[17] Bijan Saha, *Anisotropic cosmological models with a perfect fluid and a* Λ *term* (to appear in Astrophys. Space Sci.) [arXiv: gr-qc/0411080].
[18] L. D. Landau and E. M. Lifshitz, *Mechanics*, Course of Theoretical Physics No. 1 (Butterworth-Heinemann, Oxford, 1976).