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Plane-symmetric solitons of spinor and scalar fields

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We consider a system of nonlinear spinor and scalar fields with minimal coupling in general relativity. The nonlinearity in the spinor field Lagrangian is given by an arbitrary function of the invariants generated from the bilinear spinor forms $S = \bar{\psi}\psi$ and $P = \mathrm{i}\bar{\psi}\gamma^5\psi$; the scalar Lagrangian is chosen as an arbitrary function of the scalar invariant $\Omega = \varphi_{,\alpha}\varphi^{,\alpha}$, that becomes linear at $\Omega \to 0$. The spinor and the scalar fields in question interact with each other by means of a gravitational field which is given by a plane-symmetric metric. Exact plane-symmetric solutions to the gravitational, spinor and scalar field equations have been obtained. Role of gravitational field in the formation of the field configurations with limited total energy, spin and charge has been investigated. Influence of the change of the sign of energy density of the spinor and scalar fields on the properties of the configurations obtained has been examined. It has been established that under the change of the sign of the scalar field energy density the system in question can be realized physically iff the scalar charge does not exceed some critical value. In case of spinor field no such restriction on its parameter occurs. In general it has been shown that the choice of spinor field nonlinearity can lead to the elimination of scalar field contribution to the metric functions, but leaving its contribution to the total energy unaltered.

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1 Introduction

Elementary particle solitons have been extensively investigated at the level of special relativity first in the context of Abelian theories and later in non-Abelian theories. At the general relativistic level less is known. In particular, the influence

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of the gravitational field on the known special relativistic structures has not been extensively studied [1].

It is well known that the gravitational field is universal and unscreenable, i.e., it is present in microcosm. It is known as well that, at present there is no experimental data on the role of gravitational field in microcosm. So the gravitational interaction in the microcosm can be studied based on the qualitative systems only allowing exact mathematical investigations. From the relative weakness of gravitational interactions it does not follow that it can be overlooked while studying the properties of microcosm. There are theoretical indications on the fact that consideration of proper gravitational field may significantly influence the properties of interacting particles, especially at high energies.

At present the nonlinear generalization of classical field theory remains one of the possible ways to overcome the difficulties of the theory which considers elementary particles as mathematical points. In this approach elementary particles are modeled by soliton-like solutions of corresponding nonlinear equations. The gravitational field equation is nonlinear by nature and the field itself is universal and unscreenable. These properties lead to definite physical interest for the proper gravitational field to be considered. The study of the role of the gravitational field is important in its own right irrespective of the physical interpretations of the exact solutions obtained. The existence of stable particle-like classical elementary excitations in a model 5D universe was obtained in [2]. The authors showed that if a torsion invariant is included in the free Lagrangian, the particle-like stable solutions exist having definite positive rest energy, spin, and corresponding antiparticles. Solitons with spherical and/or cylindrical symmetry in the interacting system of scalar, electromagnetic, and gravitational fields were obtained in [3,4]. Nevertheless, papers dealing with soliton-like solutions of nonlinear field equations ignore the proper gravitational field in the initial field system more often than not.

The purpose of the paper is to study the role of gravitational field in the formation of extended objects in microcosm, e.g., solitons. The material fields are given by a system of nonlinear spinor and scalar ones, since the inclusion of proper gravitational field intensifies the nonlinearity of the fields in question. Within the scope of this report we show that the spinor field is more sensitive to the proper gravitational one than the scalar or electro-magnetic fields.

2 Fundamental equations and general solutions

The Lagrangian of the nonlinear spinor, scalar and gravitational fields can be written in the form

$$L = \frac{R}{2\kappa} + L_{\rm sp} + L_{\rm sc} \tag{2.1}$$

with

$$L_{\rm sp} = \frac{\rm i}{2}\left[\bar\psi\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar\psi\gamma^{\mu}\psi\right] - m\bar\psi\psi + L_{\rm N}\,, \tag{2.2}$$

and

$$L_{\rm sc} = \Psi(\Omega), \quad \Omega = \varphi_{,\alpha}\varphi^{,\alpha}. \tag{2.3}$$

Here R is the scalar curvature and κ is the Einstein's gravitational constant and ψ is the 4-component dirac spinor having the form $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^{\rm T}$, where T stands for transpose. The nonlinear term $L_{\rm N}$ in spinor Lagrangian describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field having the form

$$S = \bar\psi\psi, \quad P = {\rm i}\bar\psi\gamma^5\psi, \quad v^\mu = (\bar\psi\gamma^\mu\psi), \quad A^\mu = (\bar\psi\gamma^5\gamma^\mu\psi), \quad T^{\mu\nu} = (\bar\psi\sigma^{\mu\nu}\psi),$$

where $\sigma^{\mu\nu} = ({\rm i}/2)[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$. Invariants, corresponding to the bilinear forms, look like

$$\begin{aligned} I &= S^2, \quad J = P^2, \quad I_v = v_\mu\, v^\mu = (\bar\psi\gamma^\mu\psi)\, g_{\mu\nu}(\bar\psi\gamma^\nu\psi), \\ I_A &= A_\mu\, A^\mu = (\bar\psi\gamma^5\gamma^\mu\psi)\, g_{\mu\nu}(\bar\psi\gamma^5\gamma^\nu\psi), \\ I_T &= T_{\mu\nu}\, T^{\mu\nu} = (\bar\psi\sigma^{\mu\nu}\psi)\, g_{\mu\alpha}g_{\nu\beta}(\bar\psi\sigma^{\alpha\beta}\psi). \end{aligned}$$

According to the Pauli–Fierz theorem [5], among the five invariants only I and J are independent as all others can be expressed by them: $I_v = -I_A = I + J$ and $I_T = I - J$. Therefore we choose the nonlinear term $L_{\rm N} = F(I, J)$, thus claiming that it describes the nonlinearity in the most general of its form.

The scalar Lagrangian $L_{\rm sc}$ is an arbitrary function of invariant $\Omega = \varphi_{,\alpha}\varphi^{,\alpha}$, satisfying the condition

$$\lim_{\Omega\to 0} \Psi(\Omega) = \tfrac{1}{2}\Omega + \cdots \tag{2.4}$$

The gravitational field in our case is given by a plane-symmetric space-time. As was defined by Taub [6], a space-time will be said to have plane symmetry if it admits the three parameter group generated by the transformation

$$y^* = y + a, \tag{2.5a}$$

$$z^* = z + b, \tag{2.5b}$$

and

$$\begin{aligned} y^* &= y\cos\theta + z\sin\theta, \\ z^* &= -y\sin\theta + z\cos\theta. \end{aligned} \tag{2.5c}$$

The metric of space-time admitting plane symmetry may be written as [7]

$${\rm d}s^2 = {\rm e}^{2\chi}{\rm d}t^2 - {\rm e}^{2\alpha}{\rm d}x^2 - {\rm e}^{2\beta}({\rm d}y^2 + {\rm d}z^2), \tag{2.6}$$

where the speed of light c is taken to be unity and χ, α, β are functions of x and t alone. Note that though (2.6) is one of the most general form of a universe admitting plane symmetry, in this paper we consider the metric functions χ, α, β to be time independent and obey the coordinate condition

$$\alpha = 2\beta + \chi. \tag{2.7}$$

Let us now formulate the requirements to be fulfilled by particle-like solutions (solitons). These are [3]:

– Stationarity [applied to metric (2.6)], i.e.,

$$\chi = \chi(x), \quad \alpha = \alpha(x), \quad \beta = \beta(x).$$

– Regularity of the metric and the matter fields in the whole space-time.

– Localization in space-time (with finite energy)

$$E_f = \int T_0^0 \sqrt{-^3 g}\, \mathrm{d}V < \infty.$$

The last requirement assumes the rapid decreasing of the energy density of the material field at spatial infinity, which together with the second one guarantees the finiteness of E_{f}. The second requirement means the regularity of material fields as well as the regularity of metric functions, which entails the demand of finiteness of the energy-momentum tensor of material fields all over the space.

Variation of (2.1) with respect to spinor field $\psi\,(\bar\psi)$ gives nonlinear spinor field equations

$$\mathrm{i}\gamma^\mu \nabla_\mu \psi - \Phi\psi + \mathrm{i}\mathcal{G}\gamma^5\psi = 0, \tag{2.8a}$$
$$\mathrm{i}\nabla_\mu \bar\psi \gamma^\mu + \Phi\bar\psi - \mathrm{i}\mathcal{G}\bar\psi\gamma^5 = 0, \tag{2.8b}$$

with

$$\Phi = m - \mathcal{D} = m - 2S\frac{\partial F}{\partial I}, \qquad \mathcal{G} = 2P\frac{\partial F}{\partial J},$$

whereas, variation of (2.1) with respect to scalar field yields the following scalar field equation:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g^{\nu\mu}\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega}\varphi_{,\mu}\right) = 0. \tag{2.9}$$

Varying (2.1) with respect to metric tensor $g_{\mu\nu}$ we obtain the Einstein's field equation

$$R_\nu^\mu - \tfrac{1}{2}\,\delta_\nu^\mu R = -\kappa T_\nu^\mu\,, \tag{2.10}$$

which in view of (2.6) and (2.7) is written as follows

$$G_0^0 = \mathrm{e}^{-2\alpha}\left(2\beta'' - 2\chi'\beta' - \beta'^2\right) = -\kappa T_0^0\,, \tag{2.11a}$$
$$G_1^1 = \mathrm{e}^{-2\alpha}\left(2\chi'\beta' + \beta'^2\right) = -\kappa T_1^1\,, \tag{2.11b}$$
$$G_2^2 = \mathrm{e}^{-2\alpha}\left(\beta'' + \chi'' - 2\chi'\beta' - \beta'^2\right) = -\kappa T_2^2\,, \tag{2.11c}$$
$$G_3^3 = G_2^2, \quad T_3^3 = T_2^2. \tag{2.11d}$$

Here prime denotes differentiation with respect to x and T_ν^μ is the energy-momentum tensor of the spinor and scalar fields

$$T_\mu^\nu = T_{\mathrm{sp}\,\mu}^{\ \ \nu} + T_{\mathrm{sc}\,\mu}^{\ \ \nu}. \tag{2.12}$$

The energy-momentum tensor of the spinor field is

$$T_{\mathrm{sp}\,\mu}^{\ \ \rho} = \tfrac{1}{4}\mathrm{i}g^{\rho\nu}\left(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi\right) - \delta_\mu^\rho L_{\mathrm{sp}}\,, \tag{2.13}$$

where $L_{\rm sp}$ with respect to (2.8) takes the form

$$L_{\rm sp} = -\frac{1}{2}\left(\bar\psi\frac{\partial F}{\partial\bar\psi} + \frac{\partial F}{\partial\psi}\psi\right) - F, \tag{2.14}$$

and the energy-momentum tensor of the scalar field is

$$T_{{\rm sc}\,\mu}^{\ \ \nu} = 2\frac{{\rm d}\Psi}{{\rm d}\Omega}\varphi_{,\mu}\varphi^{,\nu} - \delta_\mu^\nu\Psi, \quad \Omega = -(\varphi')^2{\rm e}^{-2\alpha}, \quad \varphi' = \frac{{\rm d}\varphi}{{\rm d}x}. \tag{2.15}$$

In (2.8) and (2.13) ∇_μ denotes the covariant derivative of spinor, having the form [8,9]

$$\nabla_\mu\psi = \frac{\partial\psi}{\partial x^\mu} - \Gamma_\mu\psi, \tag{2.16}$$

where $\Gamma_\mu(x)$ are spinor affine connection matrices. γ matrices in the above equations are connected with the flat space-time Dirac matrices $\bar\gamma$ in the following way:

$$g_{\mu\nu}(x) = {\rm e}_\mu^a(x){\rm e}_\nu^b(x)\eta_{ab}, \quad \gamma_\mu(x) = {\rm e}_\mu^a(x)\bar\gamma_a, \tag{2.17}$$

where $\eta_{ab} = {\rm diag}(1,-1,-1,-1)$ and e_μ^a is a set of tetrad 4-vectors. Using (2.17) we obtain

$$\gamma^0(x) = {\rm e}^{-\chi}\bar\gamma^0, \quad \gamma^1(x) = {\rm e}^{-\alpha}\bar\gamma^1, \quad \gamma^2(x) = {\rm e}^{-\beta}\bar\gamma^2, \quad \gamma^3(x) = {\rm e}^{-\beta}\bar\gamma^3. \tag{2.18}$$

From

$$\Gamma_\mu(x) = \tfrac{1}{4}g_{\rho\sigma}(x)\left(\partial_\mu{\rm e}_\delta^b{\rm e}_b^\rho - \Gamma_{\mu\delta}^\rho\right)\gamma^\sigma\gamma^\delta \tag{2.19}$$

one finds

$$\Gamma_0 = -\tfrac{1}{2}\bar\gamma^0\bar\gamma^1{\rm e}^{-2\beta}\chi', \quad \Gamma_1 = 0, \quad \Gamma_2 = \tfrac{1}{2}\bar\gamma^2\bar\gamma^1{\rm e}^{-(\chi+\beta)}\beta', \quad \Gamma_3 = \tfrac{1}{2}\bar\gamma^3\bar\gamma^1{\rm e}^{-(\chi+\beta)}\beta'. \tag{2.20}$$

Flat space-time matrices $\bar\gamma$ we will choose in the form, given in [10]:

$$\bar\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \bar\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \bar\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Defining γ^5 as follows,

$$\gamma^5 = -\tfrac{1}{4}{\rm i}E_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho, \quad E_{\mu\nu\sigma\rho} = \sqrt{-g}\varepsilon_{\mu\nu\sigma\rho}, \quad \varepsilon_{0123} = 1,$$
$$\gamma^5 = -{\rm i}\sqrt{-g}\gamma^0\gamma^1\gamma^2\gamma^3 = -{\rm i}\bar\gamma^0\bar\gamma^1\bar\gamma^2\bar\gamma^3 = \bar\gamma^5,$$

we obtain

$$\bar{\gamma}^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The scalar field equation (2.9) has the solution

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega}\varphi' = \varphi_0, \quad \varphi_0 = \text{const.} \tag{2.21}$$

The equality (2.21) for a given $\Psi(\Omega)$ is an algebraic equation for φ', that is to be defined through metric function $\mathrm{e}^{\alpha(x)}$.

We will consider the spinor field to be the function of the spatial coordinate x only $[\psi = \psi(x)]$. Using (2.16), (2.18), and (2.20) we find

$$\gamma^\mu \Gamma_\mu = -\tfrac{1}{2}\mathrm{e}^{-\alpha}\alpha' \bar{\gamma}^1. \tag{2.22}$$

Then taking into account (2.22) we rewrite the spinor field equation (2.8a) as

$$\mathrm{i}\bar{\gamma}^1 \left(\frac{\partial}{\partial x} + \frac{\alpha'}{2}\right)\psi + \mathrm{i}\mathrm{e}^{\alpha}\Phi\psi + \mathrm{e}^{\alpha}\mathcal{G}\gamma^5\psi = 0. \tag{2.23}$$

Further setting $V(x) = \mathrm{e}^{\alpha/2}\psi(x)$ for the components of spinor field from (2.23) one deduces the following system of equations:

$$V_4' + \mathrm{i}\mathrm{e}^{\alpha}\Phi V_1 - \mathrm{e}^{\alpha}\mathcal{G}V_3 = 0, \tag{2.24a}$$
$$V_3' + \mathrm{i}\mathrm{e}^{\alpha}\Phi V_2 - \mathrm{e}^{\alpha}\mathcal{G}V_4 = 0, \tag{2.24b}$$
$$V_2' - \mathrm{i}\mathrm{e}^{\alpha}\Phi V_3 + \mathrm{e}^{\alpha}\mathcal{G}V_1 = 0, \tag{2.24c}$$
$$V_1' - \mathrm{i}\mathrm{e}^{\alpha}\Phi V_4 + \mathrm{e}^{\alpha}\mathcal{G}V_2 = 0. \tag{2.24d}$$

As one sees, the equation (2.24) gives following relation:

$$V_1^2 - V_2^2 - V_3^2 + V_4^2 = \text{const.} \tag{2.25}$$

Using the solutions obtained one can write the components of spinor current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{2.26}$$

Taking into account that $\bar{\psi} = \psi^\dagger\bar{\gamma}^0$, where $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ and $\psi_j = \mathrm{e}^{-\alpha/2}V_j,\ j = 1,2,3,4$, for the components of spinor current we write

$$j^0 = (V_1^* V_1 + V_2^* V_2 + V_3^* V_3 + V_4^* V_4)\,\mathrm{e}^{-(\alpha+\chi)}, \tag{2.27a}$$
$$j^1 = (V_1^* V_4 + V_2^* V_3 + V_3^* V_2 + V_4^* V_1)\,\mathrm{e}^{-2\alpha}, \tag{2.27b}$$
$$j^2 = -\mathrm{i}\,(V_1^* V_4 - V_2^* V_3 + V_3^* V_2 - V_4^* V_1)\,\mathrm{e}^{-(\alpha+\beta)}, \tag{2.27c}$$
$$j^3 = (V_1^* V_3 - V_2^* V_4 + V_3^* V_1 - V_4^* V_2)\,\mathrm{e}^{-(\alpha+\beta)}. \tag{2.27d}$$

Since we consider the field configuration to be a static one, the spatial components of spinor current vanishes, i.e.,

$$j^1 = 0, \quad j^2 = 0, \quad j^3 = 0. \tag{2.28}$$

This assumption gives additional relation between the constant of integration. The component j^0 defines the charge density of spinor field that has the following chronometric-invariant form

$$\varrho = (j_0 \cdot j^0)^{1/2}. \tag{2.29}$$

The total charge of spinor field is defined as

$$Q = \int \varrho \sqrt{-{}^3g}\, \mathrm{d}V, \quad \mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{2.30a}$$

Since $\varrho = \varrho(x)$, i.e., matter distribution takes place along x axis only, for Q to make any sense we should integrate it for any finite range by y and z and then normalize it to unity. Sometimes Q is defined as

$$Q = \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \varrho\sqrt{-{}^3g}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}z}. \tag{2.30b}$$

In what follows we perform integration by y and z in the limit (0, 1) and define the total charge (normalized) as

$$Q = \int\limits_{-\infty}^{\infty} \varrho\sqrt{-{}^3g}\,\mathrm{d}x. \tag{2.30c}$$

Let us consider the spin tensor [10]

$$S^{\mu\nu,\epsilon} = \tfrac{1}{4}\bar\psi \left\{\gamma^\epsilon\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\epsilon\right\}\psi. \tag{2.31}$$

We write the components $S^{ik,0}$ $(i,k = 1,2,3)$, defining the spatial density of spin vector explicitly. From (2.31) we have

$$S^{ij,0} = \tfrac{1}{4}\bar\psi\left\{\gamma^0\sigma^{ij} + \sigma^{ij}\gamma^0\right\}\psi = \tfrac{1}{2}\bar\psi\gamma^0\sigma^{ij}\psi \tag{2.32}$$

that defines the projection of spin vector on k axis. Here i, j, k takes the value 1, 2, 3 and $i \neq j \neq k$. Thus, for the projection of spin vectors on the X, Y and Z axis we find

$$S^{23,0} = (V_1^*V_2 + V_2^*V_1 + V_3^*V_4 + V_4^*V_3)\,\mathrm{e}^{-\alpha-2\beta-\chi}, \tag{2.33a}$$
$$S^{31,0} = (V_1^*V_2 - V_2^*V_1 + V_3^*V_4 - V_4^*V_3)\,\mathrm{e}^{-2\alpha-\beta-\chi}, \tag{2.33b}$$
$$S^{12,0} = (V_1^*V_1 - V_2^*V_2 + V_3^*V_3 - V_4^*V_4)\,\mathrm{e}^{-2\alpha-\beta-\chi}. \tag{2.33c}$$

The chronometric invariant spin tensor takes the form

$$S_{\rm ch}^{ij,0} = \left(S_{ij,0} S^{ij,0}\right)^{1/2} \tag{2.34}$$

and the projection of the spin vector on k axis is defined by

$$S_k = \int\limits_{-\infty}^{\infty} S_{\rm ch}^{ij,0} \sqrt{-{}^3 g}\, {\rm d}x. \tag{2.35}$$

(In (2.35), as well as in (2.30c) integrations by y and z are performed in the limit $(0,1)$).

From (2.8) one can write the equations for $S = \bar\psi\psi$, $\quad P = {\rm i}\bar\psi\gamma^5\psi$, and $A = \bar\psi\bar\gamma^5\bar\gamma^1\psi$:

$$S' + \alpha' S + 2{\rm e}^{\alpha}\mathcal{G}\, A = 0, \tag{2.36a}$$

$$P' + \alpha' P + 2{\rm e}^{\alpha}\varPhi\, A = 0, \tag{2.36b}$$

$$A' + \alpha' A + 2{\rm e}^{\alpha}\varPhi\, P + 2{\rm e}^{\alpha}\mathcal{G} S = 0. \tag{2.36c}$$

Note that A in (2.36) is indeed the pseudo-vector A^1. Here for simplicity, we use the notation A. From (2.36) immediately follows

$$S^2 + P^2 - A^2 = C_0 {\rm e}^{-2\alpha}, \quad C_0 = {\rm const.} \tag{2.37}$$

Let us now solve the Einstein equations. To do it we first write the expression for the components of the energy-momentum tensor explicitly. Using the property of flat space-time Dirac matrices and the explicit form of covariant derivative ∇_μ, for the spinor field one finds

$$T_{{\rm sp}1}^{\ 1} = m\, S - F(I,\, J), \quad T_{{\rm sp}0}^{\ 0} = T_{{\rm sp}2}^{\ 2} = T_{{\rm sp}3}^{\ 3} = \mathcal{D}S + \mathcal{G}P - F(I,\, J). \tag{2.38}$$

On the other hand, taking into account that the scalar field φ is also a function of x only $[\varphi = \varphi(x)]$ for the scalar field one obtains

$$T_{{\rm sc}1}^{\ 1} = 2\varOmega \frac{{\rm d}\varPsi}{{\rm d}\varOmega} - \varPsi(\varOmega), \quad T_{{\rm sc}0}^{\ 0} = T_{{\rm sc}2}^{\ 2} = T_{{\rm sc}3}^{\ 3} = -\varPsi(\varOmega). \tag{2.39}$$

In view of $T_0^0 = T_2^2$, subtraction of Einstein equations (2.11a) and (2.11c) leads to the equation

$$\beta'' - \chi'' = 0 \tag{2.40}$$

with the solution

$$\beta(x) = \chi(x) + Bx, \tag{2.41}$$

where B is the integration constant. The second constant has been chosen to be trivial, since it acts on the scale of Y and Z axes only. In account of (2.40) from (2.7) one obtains

$$\beta'' = \tfrac{1}{3}\alpha'', \quad \chi'' = \tfrac{1}{3}\alpha''. \tag{2.42}$$

Solutions to the equation (2.42) together with (2.7) and (2.41) lead to the following expression for $\beta(x)$ and $\gamma(x)$:

$$\beta(x) = \tfrac{1}{3}\left[\alpha(x) + BX\right], \quad \chi(x) = \tfrac{1}{3}\left[\alpha(x) - 2Bx\right]. \tag{2.43}$$

Equation (2.11b), being the first integral of (2.11a) and (2.11c), is a first order differential equation. Inserting β and γ from (2.43) and T_1^1 in account of (2.12), (2.38), and (2.39) into (2.11b) for α, one gets

$$\alpha'^2 - B^2 = -3\kappa e^{2\alpha}\left[mS - F(I,J) + 2\Omega\frac{d\Psi}{d\Omega} - \Psi(\Omega)\right]. \tag{2.44}$$

As one sees from (2.36) and (2.37), the invariants are the functions of α, so is the right hand side of (2.44), hence can be solved in quadrature:

$$\int\frac{d\alpha}{\sqrt{B^2 - 3\kappa e^{2\alpha}\left[mS - F(I,J) + 2\Omega\frac{d\Psi}{d\Omega} - \Psi(\Omega)\right]}} = x. \tag{2.45}$$

In the sections to follow, we analyze the equation (2.44) in details given the particular form of nonlinear term in spinor Lagrangian.

3 Analysis of the results

In this section we shall analyze the general results obtained in the previous section for concrete nonlinear term.

A. Case with linear spinor and scalar fields

Let us consider the self-consistent system of linear spinor and massless scalar field equations. By doing so we can compare the results obtained with those of the self-consistent system of nonlinear spinor and scalar field equations, hence clarify the role of nonlinearity of the fields in question in the formation of regular localized solutions such as static solitary wave or solitons [11, 12].

In this case for the scalar field we have $\Psi(\Omega) = \frac{1}{2}\Omega$. Inserting this into (2.21) we obtain

$$\varphi'(x) = \varphi_0. \tag{3.1}$$

From (2.39) in account of (3.1) we get

$$-T_{\rm sc1}^{\ 1} = T_{\rm sc0}^{\ 0} = T_{\rm sc2}^{\ 2} = T_{\rm sc3}^{\ 3} = -\tfrac{1}{2}\Omega = \tfrac{1}{2}\varphi_0^2 e^{-2\alpha}. \tag{3.2}$$

On the other hand for the linear spinor field we have

$$T_{\rm sp1}^{\ 1} = m\,S, \quad T_{\rm sp0}^{\ 0} = T_{\rm sp2}^{\ 2} = T_{\rm sp3}^{\ 3} = 0. \tag{3.3}$$

As one can easily verify, for the linear spinor field the equation (2.36a) results in

$$S = C_0 e^{-\alpha}. \tag{3.4}$$

Taking this relation into account and the fact that $\alpha'(x) = -(1/S)\mathrm{d}S/\mathrm{d}x$ from (2.44) we write

$$\int \frac{\mathrm{dS}}{\sqrt{(1+\frac{1}{2}\bar{\kappa})B^2S^2 - 3\kappa C_0^2 S}} = x, \quad \bar{\kappa} = 3\kappa\frac{\varphi_0^2}{B^2}, \tag{3.5}$$

with the solution

$$S(x) = \frac{M^2}{H^2}\cosh^2(\tilde{H}x), \quad M^2 = 3\kappa C_0^2, \quad H^2 = B^2(1+\tfrac{1}{2}\bar{\kappa}), \quad \tilde{H} = \tfrac{1}{2}H. \tag{3.6}$$

Further we define the functions ψ_j. Taking into account that in this case

$$\mathcal{F}(S) = \frac{mC_0}{S\sqrt{H^2S^2 - M^2S}},$$

for $N_{1,2}$ in view of (3.6) we find

$$N_{1,2}(x) = \pm\frac{2H}{3\kappa C_0}\tanh(\tilde{H}x) + R_{1,2}.$$

We can then finally write

$$\begin{aligned} \psi_{1,2}(x) &= \mathrm{i}a_{1,2}E(x)\cosh[f(x) + R_{1,2}], \\ \psi_{3,4}(x) &= a_{2,1}E(x)\sinh[f(x) + R_{2,1}], \end{aligned} \tag{3.7}$$

where $E(x) = \sqrt{3\kappa m C_0/H^2}\cosh(\tilde{H}x)$ and $f(x) = (2H/3\kappa C_0)\tanh(\tilde{H}x)$. For the scalar field energy density we find

$$T_{\mathrm{sc}0}^{\ 0}(x) = \frac{1}{2}\varphi_0^2\mathrm{e}^{-2\alpha} = \frac{M^4\varphi_0^2}{2C_0^2H^4}\cosh^4(\tilde{H}x). \tag{3.8}$$

It is clear from (3.8) that the scalar field energy density is not localized.

Let us consider the case when the scalar field possesses negative energy density. Then we have $\Psi(\Omega) = -(1/2)\Omega$ and

$$-T_{\mathrm{sc}1}^{\ 1} = T_{\mathrm{sc}0}^{\ 0} = T_{\mathrm{sc}2}^{\ 2} = T_{\mathrm{sc}3}^{\ 3} = \tfrac{1}{2}\Omega = -\tfrac{1}{2}\varphi_0^2\mathrm{e}^{-2\alpha}. \tag{3.9}$$

Then for S we get

$$\int \frac{\mathrm{d}S}{\sqrt{(1-\frac{1}{2}\bar{\kappa})B^2S^2 - 3\kappa C_0^2 S}} = x. \tag{3.10}$$

As one sees, the field system considered here is physically realizable iff $1 - \bar{\kappa}/2 > 0$, i.e., the scalar charge $|\varphi_0| < \sqrt{2/3\kappa}B$. Moreover, in the specific case with $B = 0$, independent to the quantity of scalar charge φ_0, the existence of scalar field with negative energy density in general relativity is impossible (even in absence of linear spinor field).

For the total charge Q of the system in this case we have

$$Q = 2a^2 \int\limits_{-\infty}^{\infty} \cosh\left(\frac{4H}{3\kappa C_0}\tanh(\tilde{H}x) + 2R\right)\left(\frac{C_0 H^2}{M^2 \cosh^2(\tilde{H}x)}\right)^{3/2} \mathrm{e}^{2Bx/3}\,\mathrm{d}x < \infty. \tag{3.11}$$

It can be shown that, in case of linear spinor and scalar fields with minimal coupling both charge and spin of spinor field are limited. The energy density of the system, in view of (3.3) is defined by the contribution of scalar field only:

$$T_0^0(x) = T_{\mathrm{sc}0}^{\ 0}(x) = \frac{1}{2}\frac{\varphi_0^2 M^4}{C_0^2 H^4}\cosh^4(\tilde{H}x). \tag{3.12}$$

From (3.12) follows that, the energy density of the system is not localized and the total energy of the system $E = \int\limits_{-\infty}^{\infty} T_0^0\sqrt{-{}^3g}\,\mathrm{d}x$ is not finite.

B. Nonlinear spinor and linear scalar fields

In this subsection we study the system of nonlinear spinor and linear scalar fields together with a plane-symmetric gravitational one. As a nonlinear spinor field Lagrangian we consider the following cases.

Case I: $F = F(I)$

Let us consider the case when the nonlinear term in spinor field Lagrangian is a function of I (S^2) only. This assumption leads to $\mathcal{G} = 0$. From (2.36) as in case of linear spinor field we find $S = C_0\mathrm{e}^{-\alpha(x)}$. Proceeding as in foregoing subsection, for S from (2.44) we write

$$\frac{\mathrm{d}S}{\mathrm{d}x} = \pm\mathcal{L}(S), \quad \mathcal{L}(S) = \sqrt{B^2S^2 - 3\kappa C_0^2\left[mS - F(S) + 2\Omega\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} - \Psi(\Omega)\right]} \tag{3.13}$$

with the solution

$$\int \frac{\mathrm{d}S}{\mathcal{L}(S)} = \pm(x + x_0). \tag{3.14}$$

Given the concrete form of the functions $F(S)$ and $\Psi(\Omega)$, from (3.14) yields S, hence α, β, χ.

Let us now go back to spinor field equations (2.24). Setting $V_j(x) = U_j(S)$, $j = 1, 2, 3, 4$, and taking into account that in this case $\mathcal{G} = 0$, for $U_j(S)$ we obtain

$$\frac{\mathrm{d}U_4}{\mathrm{d}S} + \mathrm{i}\mathcal{F}(S)U_1 = 0, \tag{3.15a}$$

$$\frac{\mathrm{d}U_3}{\mathrm{d}S} + \mathrm{i}\mathcal{F}(S)U_2 = 0, \tag{3.15b}$$

$$\frac{\mathrm{d}U_2}{\mathrm{d}S} - \mathrm{i}\mathcal{F}(S)U_3 = 0, \tag{3.15c}$$

$$\frac{\mathrm{d}U_1}{\mathrm{d}S} - \mathrm{i}\mathcal{F}(S)U_4 = 0, \tag{3.15d}$$

with $\mathcal{F}(S) = \Phi\mathcal{L}(S)C_0/S$. Differentiating (3.15a) with respect to S and inserting (3.15d) into it for U_4 we find

$$\frac{\mathrm{d}^2U_4}{\mathrm{d}S^2} - \frac{1}{\mathcal{F}}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}S}\frac{\mathrm{d}U_4}{\mathrm{d}S} - \mathcal{F}^2U_4 = 0 \tag{3.16}$$

that transforms to

$$\frac{1}{\mathcal{F}}\frac{\mathrm{d}}{\mathrm{d}S}\left(\frac{1}{\mathcal{F}}\frac{\mathrm{d}U_4}{\mathrm{d}S}\right) - U_4 = 0, \tag{3.17}$$

with the first integral

$$\frac{\mathrm{d}U_4}{\mathrm{d}S} = \pm\sqrt{U_4^2 + C_1}\cdot\mathcal{F}(S), \quad C_1 = \mathrm{const.} \tag{3.18}$$

For $C_1 = a_1^2 > 0$ from (3.18) we obtain

$$U_4(S) = a_1 \sinh N_1(S), \quad N_1 = \pm\int\mathcal{F}(S)\,\mathrm{d}S + R_1, \quad R_1 = \mathrm{const.}, \tag{3.19}$$

whereas, for $C_1 = -b_1^2 < 0$ from (3.18) we obtain

$$U_4(S) = a_1 \cosh N_1(S)\,. \tag{3.20}$$

Inserting (3.19) and (3.20) into (3.15d) one finds

$$U_1(S) = \mathrm{i}a_1\cosh N_1(S), \quad U_1(S) = \mathrm{i}b_1\sinh N_1(S). \tag{3.21}$$

Analogically, for U_2 and U_3 we obtain

$$U_3(S) = a_2\sinh N_2(S), \quad U_3(S) = b_2\cosh N_2(S) \tag{3.22}$$

and

$$U_2(S) = \mathrm{i}a_2\cosh N_2(S), \quad U_2(S) = \mathrm{i}b_2\sinh N_2(S)\,, \tag{3.23}$$

where $N_2 = \pm\int\mathcal{F}(S)\mathrm{d}S + R_2$ and a_2, b_2, and R_2 are the integration constants. Thus we find the general solutions to the spinor field equations (3.15) containing four arbitrary constants.

Using the solutions obtained, from (2.27) we find the components of spinor current

$$j^0 = \left[a_1^2\cosh(2N_1(S)) + a_2^2\cosh(2N_2(S))\right]\mathrm{e}^{-(\alpha+\chi)}, \tag{3.24a}$$
$$j^1 = 0, \tag{3.24b}$$
$$j^2 = -\left[a_1^2\sinh(2N_1(S)) - a_2^2\sinh(2N_2(S))\right]\mathrm{e}^{-(\alpha+\beta)}, \tag{3.24c}$$
$$j^3 = 0. \tag{3.24d}$$

The supposition (2.28) leads to the following relations between the constants: $a_1 = a_2 = a$ and $R_1 = R_2 = R$, since $N_1(S) = N_2(S) = N(S)$. The chronometric-invariant form of the charge density and the total charge of spinor field are

$$\varrho = 2a^2\cosh(2N(S))\mathrm{e}^{-\alpha}, \tag{3.25}$$

$$Q = 2a^2 \int\limits_{-\infty}^{\infty} \cosh(2N(S)) \mathrm{e}^{\alpha-\chi} \mathrm{d}x. \tag{3.26}$$

From (2.32) we find

$$S^{12,0} = 0, \quad S^{13,0} = 0, \quad S^{23,0} = a^2 \cosh(2N(S)) \mathrm{e}^{-2\alpha}. \tag{3.27}$$

Thus, the only nontrivial component of the spin tensor is $S^{23,0}$ that defines the projection of spin vector on X axis. From (2.34) we write the chronometric invariant spin tensor

$$S_{\mathrm{ch}}^{23,0} = a^2 \cosh(2N(S)) \mathrm{e}^{-\alpha}, \tag{3.28}$$

and the projection of the spin vector on X axis

$$S_1 = a^2 \int\limits_{-\infty}^{\infty} \cosh(2N(S)) \mathrm{e}^{\alpha-\chi} \mathrm{d}x \tag{3.29}$$

(in (2.35), as well as in (2.30c) integrations by y and z are performed in the limit $(0,1)$). Note that the integrands in (3.26) and (3.29) coincide.

Let us now analyze the result obtained choosing the nonlinear term in the form $F(I) = \lambda S^n = \lambda I^{n/2}$ with $n \geq 2$ and λ is the parameter of nonlinearity. For $n = 2$ we have Heisenberg–Ivanenko type nonlinear spinor field equation [13]

$$\mathrm{i}\mathrm{e}^{-\alpha} \bar{\gamma}^1 \left(\partial_x + \tfrac{1}{2}\alpha'\right) \psi - m\psi + 2\lambda(\bar{\psi}\psi)\psi = 0. \tag{3.30}$$

Setting $F = S^2$ into (3.14) we come to the expression for S that is similar to that for linear case with

$$H^2 \to H_1^2 = B^2 + 3\kappa\lambda C_0 + \tfrac{3}{2}\kappa\varphi_0^2. \tag{3.31}$$

Let us write the functions ψ_j explicitly. In this case we have

$$\mathcal{F}(S) = \frac{m(C_0 - 2\lambda S)}{S\sqrt{H_1^2 S^2 - M^2 S}}$$

and

$$N_{1,2}(x) = \frac{2H_1}{3\kappa C_0} \tanh(\bar{H}_1 x) - 2\lambda C_0 x + R_{1,2}, \quad \bar{H}_1 = \tfrac{1}{2} H_1.$$

We can then finally write

$$\begin{aligned} \psi_{1,2}(x) &= \mathrm{i}a_{1,2} \frac{\sqrt{3\kappa m C_0}}{H_1} \cosh(\bar{H}_1 x) \cosh N_{1,2}(x), \\ \psi_{3,4}(x) &= \mathrm{i}a_{2,1} \frac{\sqrt{3\kappa m C_0}}{H_1} \cosh(\bar{H}_1 x) \cosh N_{2,1}(x). \end{aligned} \tag{3.32}$$

Let us consider the energy-density distribution of the field system:

$$T_0^0 = \left(\lambda + \frac{1}{2}\frac{\varphi_0^2}{C_0^2}\right) \frac{M^4}{H_1^4} \cosh^4(\bar{H}_1 x). \tag{3.33}$$

From (3.33) follows that, the energy density of the system is not localized and the total energy of the system $E = \int\limits_{-\infty}^{\infty} T_0^0 \sqrt{-^3 g}\,\mathrm{d}x$ is not finite. Note that, the energy density of the system can be trivial, if

$$\lambda + \frac{1}{2}\frac{\varphi_0^2}{C_0^2} = 0. \tag{3.34}$$

It is possible, iff the sign of energy density of spinor and scalar fields are different.

Let us write the total charge of the system.

$$Q = 2a^2 \int\limits_{-\infty}^{\infty} \cosh\left(\frac{4H_1}{3\kappa C_0}\tanh(\bar{H}_1 x) - 4\lambda C_0 x + 2R\right)\left(\frac{C_0 H_1^2}{M^2 \cosh^2(\bar{H}_1 x)}\right)^{3/2} \mathrm{e}^{2Bx/3}\,\mathrm{d}x. \tag{3.35}$$

If $12\lambda^2 C_0^2 + \lambda C_0(4B - \kappa C_0) - \kappa\varphi_0^2/2 < 0$, the integral (3.35) converges, that means the possibility of existence of finite charge and spin of the system.

In case of $n > 2$, the energy density of the system in question is

$$T_0^0 = \lambda(n-1)S^n + \frac{1}{2}\frac{\varphi_0^2}{C_0^2}S^2, \tag{3.36}$$

which shows that the regular solutions with localized energy density exists iff $S = \bar{\psi}\psi$ is a continuous and limited function and $\lim\limits_{x\to\pm\infty} S(x) \to 0$. The condition, when S possesses the properties mentioned above, is

$$\int \frac{\mathrm{d}S}{\sqrt{(1+\frac{1}{2}\bar{\kappa})B^2S^2 - 3\kappa C_0^2(mS - \lambda S^n)}} = x. \tag{3.37}$$

As one sees from (3.37), for $m \neq 0$ at no value of x S becomes trivial, since as $S \to 0$, the denominator of the integrant beginning from some finite value of S becomes imaginary. It means that for $S(x)$ to be trivial at spatial infinity ($x \to \infty$), it is necessary to choose massless spinor field setting $m = 0$ in (3.37). It should be emphasized that, in the unified nonlinear spinor theory of Heisenberg, the massive term is absent and, according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [14]. Indeed, in the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system [15]. Thus without losing the generality we can consider massless spinor field putting $m = 0$.

From (3.37) for $m = 0$, $\lambda > 0$ and $n > 2$ for $S(x)$ we obtain

$$S(x) = \left[-H_1/\sqrt{3\kappa\lambda C_0^2(\zeta^2 - 1)}\right]^{2/(n-2)}, \quad \zeta = \cosh[(n-2)\bar{H}_1 x]\,, \tag{3.38}$$

from which follows that $\lim\limits_{x\to 0} |S(x)| \to \infty$. It means that $T_0^0(x)$ is not bounded at $x = 0$ and the initial system of equations does not possess solutions with localized energy density.

If we set in (3.37) $m = 0$, $\lambda = -\Lambda^2 < 0$, and $n > 2$, then for S we obtain

$$S(x) = \left(\frac{H_1}{\sqrt{3\kappa\lambda C_0^2 \zeta}}\right)^{2/(n-2)}. \tag{3.39}$$

It is seen from (3.39) that $S(x)$ has maximum at $x = 0$ and $\lim\limits_{x\to\pm\infty} S(x) \to 0$. For energy density we have

$$T_0^0 = -\Lambda^2(n-1)S^n + \frac{1}{2}\frac{\varphi_0^2}{C_0^2}S^2, \tag{3.40}$$

where S is defined by (3.39). In view of S it follows that $T_0^0(x)$ is an alternating function.

Let us find the condition when the total energy of the system is bound by

$$E = \int\limits_{-\infty}^{\infty} T_0^0 \sqrt{-^3 g}\, \mathrm{d}x < \infty. \tag{3.41}$$

For this we write the integrand of (3.41):

$$\begin{aligned} \varepsilon(x) &= T_0^0 \sqrt{-^3 g} = \\ &= C_0^{5/3}\left(\frac{\varphi_0^2}{2C_0^2} - \frac{(n-1)H_1^2\zeta^2}{3\kappa\lambda C_0^2}\right)\left(\frac{H_1^2\zeta}{3\kappa\Lambda^2 C_0^2}\right)^{1/3(n-2)} \mathrm{e}^{2Bx/3}. \end{aligned} \tag{3.42}$$

From (3.42) it follows that $\lim\limits_{x\to-\infty} \varepsilon(x) \to 0$ for any value of the parameters, while $\lim\limits_{x\to+\infty} \varepsilon(x) \to 0$ iff $H > 2B$ or $\kappa\varphi_0^2 > 2B^2$. Contribution of scalar field to the total energy in this case is positive and finite:

$$T_{\mathrm{sc}0}^{\ 0} = \frac{\varphi_0^2}{2C_0^2}S^2, \quad E_{\mathrm{sc}} = \int\limits_{-\infty}^{\infty} T_{\mathrm{sc}0}^{\ 0} \sqrt{-^3 g}\, \mathrm{d}x < \infty. \tag{3.43}$$

Let us remark that in the case considered the scalar field is linear and massless. As far as in absence of spinor field energy density of the linear scalar field is not localized and the total energy in not finite, in the case considered the properties of the field configurations are defined by those of nonlinear spinor field. The contribution of nonlinear spinor field to the total energy is negative. Moreover, it remains finite even in absence of scalar field for $n > 2$ [16].

The components of spinor field in this case have the form

$$\begin{aligned} \psi_{1,2}(x) &= \mathrm{i}a_{1,2}E(x)\cosh N_{1,2}(x), \\ \psi_{3,4}(x) &= a_{2,1}E(x)\sinh N_{2,1}(x), \end{aligned} \tag{3.44}$$

where

$$E(x) = \frac{1}{\sqrt{C_0}}\left(\frac{H_1}{\sqrt{3\kappa\Lambda^2 C_0^2\zeta}}\right)^{1/(n-2)}$$

and

$$N_{1,2}(x) = -\frac{2nH_1\sqrt{\zeta^2-1}}{3\kappa C_0(n-2)\zeta} + R_{1,2}.$$

For the solutions obtained we write the chronometric-invariant charge density of the spinor field ϱ:

$$\varrho(x) = \frac{2a^2}{C_0}\cosh\left(-\frac{4nH_1\sqrt{\zeta^2-1}}{3\kappa C_0(n-2)\zeta} + 2R\right)\left(\frac{H_1^2}{3\kappa\Lambda^2 C_0^2\zeta^2}\right)^{1/(n-2)}. \tag{3.45}$$

As one sees from (3.45), the charge density is localized, since $\lim\limits_{x\to\pm\infty}\varrho(x) \to 0$. Nevertheless, the charge density of the spinor field, coming to unit invariant volume $\varrho\sqrt{-^3g}$, is not localized:

$$\varrho\sqrt{-^3g} = 2a^2\cosh[2N(x)]\mathrm{e}^{\alpha-\gamma} = 2a^2\cosh[2N(x)]\left(\frac{C_0}{S}\right)^{2/3}\mathrm{e}^{2Bx/3}. \tag{3.46}$$

It leads to the fact that the total charge of the spinor field is not bounded as well. As far as the expression for chronometric-invariant tensor of spin (3.28) coincides with that of $\varrho(x)/2$, the conclusions made for $\varrho(x)$ and Q will be valid for the spin tensor $S^{23,0}_{\rm ch}$ and projection of spin vector on X axis S_1, i.e., $S^{23,0}_{\rm ch}$ is localized and S_1 is unlimited.

The solution obtained describes the configuration of nonlinear spinor and linear scalar fields with localized energy density but with the metric that is singular at spatial infinity, as in this case

$$\mathrm{e}^{2\alpha} = \left(\frac{C_0}{S}\right)^2 = C_0^2\left(\frac{3\kappa\Lambda C_0^2\zeta}{H_1^2}\right)^{2/(n-2)}\Big|_{x\to\pm\infty} \to \infty\ . \tag{3.47}$$

Let us consider the massless spinor field with

$$F = -\Lambda^2 S^{-\nu}, \quad \nu = \text{const.} > 0. \tag{3.48}$$

In this case the energy density of the system of nonlinear spinor and linear scalar fields with minimal coupling takes the form

$$T_0^0 = \Lambda^2(\nu+1)S^{-\nu} + \frac{\varphi_0^2}{2C_0^2}S^2\,. \tag{3.49}$$

For S in this case we get

$$\int\frac{\mathrm{d}S}{\sqrt{(1+\frac{1}{2}\bar\kappa)B^2S^2 - 3\kappa C_0^2\Lambda^2 S^{-\nu}}} = x \tag{3.50}$$

with the solution

$$S(x) = \left(\frac{3\kappa\Lambda^2 C_0^2}{H_1^2}\zeta_1^2\right)^{1/(\nu+2)}, \quad \zeta_1 = \cosh[(\nu+2)\bar H_1 x]. \tag{3.51}$$

For energy density in this case we have

$$T_0^0(x) = \Lambda^2(\nu+1)\left(\frac{H_1^2}{3\kappa C_0^2\Lambda^2\zeta_1^2}\right)^{\nu/(\nu+2)} + \frac{\varphi_0^2}{2C_0^2}\left(\frac{3\kappa C_0^2\Lambda^2\zeta_1^2}{H_1^2}\right)^{2/(\nu+2)}. \quad (3.52)$$

It follows from (3.52) that the contribution of the spinor field in the energy density is localized while for the scalar field it is not the case.

The energy density distribution of the field system, coming to unit invariant volume is

$$\begin{aligned}\varepsilon(x) &= T_0^0\sqrt{-{}^3g} = \left[\Lambda^2(\nu+1)S^{-\nu} + \frac{\varphi_0^2}{2C_0^2}S^2\right]\mathrm{e}^{2\alpha-\gamma} \\ &= \left(\frac{H_1^2(\nu+1)}{3\kappa\zeta_1^2} + \frac{\varphi_0^2}{2}\right)\left(\frac{H_1^2}{3\kappa C_0^2\Lambda^2\zeta_1^2}\right)^{1/3(\nu+2)}\mathrm{e}^{2Bx/3}. \end{aligned} \quad (3.53)$$

As one sees from (3.53), $\varepsilon(x)$ is a localized function, i.e., $\lim_{x\to\pm\infty}\varepsilon(x)\to 0$, if $H > 2B$ or $\kappa\varphi_0^2 > 2B^2$. In this case the total energy is also finite.

The components of spinor field in this case have the form

$$\begin{aligned}\psi_{1,2}(x) &= \mathrm{i}a_{1,2}E(x)\cosh N_{1,2}(x),\\ \psi_{3,4}(x) &= a_{2,1}E(x)\sinh N_{2,1}(x),\end{aligned} \quad (3.54)$$

where

$$E(x) = \frac{1}{\sqrt{C_0}}\left(\frac{\sqrt{3\kappa\Lambda^2C_0^2}}{H_1^2}\zeta_1\right)^{1/(\nu+2)}$$

and

$$N_{1,2}(x) = -\frac{2H\nu\sqrt{\zeta_1^2-1}}{3\kappa C_0(\nu+2)\zeta_1} + R_{1,2}.$$

The chronometric-invariant charge density of the spinor field coming to unit invariant volume with $a_1 = a_2 = a$ and $N_1 = N_2$ reads

$$\begin{aligned}\varrho\sqrt{-{}^3g} &= 2a^2\cosh[2N(x)]\mathrm{e}^{\alpha-\gamma} = \\ &= 2a^2(C_0)^{2/3}\cosh\left(2R - \frac{4H_1\nu\sqrt{\zeta_1^2-1}}{3\kappa C_0(\nu+2)\zeta_1}\right)\left(\frac{H_1^2}{3\kappa C_0^2\Lambda^2\zeta_1^2}\right)^{2/3(\nu+2)}\mathrm{e}^{2Bx/3}.\end{aligned} \quad (3.55)$$

It follows from (3.55) that $\varrho\sqrt{-{}^3g}$ is a localized function and the total charge Q is finite. The spin of spinor field is limited as well.

Case II: $F = F(J)$

Here we consider the massless spinor field with the nonlinear term being a function of J alone, i.e., $F = F(J)$. In this case from (2.36b) immediately follows the relation

$$P = D_0\mathrm{e}^{-\alpha(x)}, \quad D_0 = \text{const}. \quad (3.56)$$

For the components of the spinor field from (2.24) one finds

$$V_4' - \mathrm{e}^{\alpha}\mathcal{G}V_3 = 0, \tag{3.57a}$$
$$V_3' - \mathrm{e}^{\alpha}\mathcal{G}V_4 = 0, \tag{3.57b}$$
$$V_2' + \mathrm{e}^{\alpha}\mathcal{G}V_1 = 0, \tag{3.57c}$$
$$V_1' + \mathrm{e}^{\alpha}\mathcal{G}V_2 = 0. \tag{3.57d}$$

Defining $\mathcal{A} = \int \mathrm{e}^{\alpha}\mathcal{G}\mathrm{d}x$ solutions of (3.57) can be written as

$$V_1 = D_1 \sinh(-\mathcal{A} + D_2), \tag{3.58a}$$
$$V_2 = D_1 \cosh(-\mathcal{A} + D_2), \tag{3.58b}$$
$$V_3 = D_3 \sinh(\mathcal{A} + D_4), \tag{3.58c}$$
$$V_4 = D_3 \cosh(\mathcal{A} + D_4), \tag{3.58d}$$

with D_1, D_2, D_3, and D_4 being the constants of integration.

Using the solutions obtained, from (2.27) we now find the components of spinor current

$$j^0 = \left\{C_1^2 \cosh[2(-\mathcal{A} + C_2)] + C_3^2 \cosh[2(\mathcal{A} + C_4)]\right\} \mathrm{e}^{-(\alpha+\chi)}, \tag{3.59a}$$
$$j^1 = [2C_1C_3 \sinh(C_2 + C_4)]\, \mathrm{e}^{-2\alpha}, \tag{3.59b}$$
$$j^2 = 0, \tag{3.59c}$$
$$j^3 = -\,[2C_1C_3 \cosh(2\mathcal{A} - C_2 + C_4)]\, \mathrm{e}^{-(\alpha+\beta)}. \tag{3.59d}$$

The supposition (2.28) that the spatial components of the spinor current are trivial leads at least one of the constants (D_1, D_3) to be zero. Let us set $D_1 = 0$. The chronometric-invariant form of the charge density and the total charge of spinor field are

$$\varrho = D_3^2 \cosh[2(\mathcal{A} + D_4)]\mathrm{e}^{-\alpha}, \tag{3.60}$$

$$Q = D_3^2 \int_{-\infty}^{\infty} \cosh[2(\mathcal{A} + D_4)]\mathrm{e}^{\alpha-\chi}\mathrm{d}x. \tag{3.61}$$

For the components of the spin tensor from (2.32) we find

$$S^{12,0} = -D_3^2 \mathrm{e}^{-(2\alpha+\beta+\chi)}, \quad S^{31,0} = 0, \quad S^{23,0} = D_3^2 \sinh[2(\mathcal{A} + D_4)]\mathrm{e}^{-2\alpha}. \tag{3.62}$$

Thus, in this case we have two nontrivial components of the spin tensor $S^{23,0}$ and $S^{12,0}$. those define the projections of spin vector on X and Z axis, respectively. From (2.34) we write the chronometric invariant spin tensor

$$S_{\mathrm{ch}}^{23,0} = D_3^2 \sinh\left[2(\mathcal{A} + D_4)\right] \mathrm{e}^{-\alpha}, \tag{3.63a}$$
$$S_{\mathrm{ch}}^{23,0} = D_3^2 \mathrm{e}^{-\alpha} \tag{3.63b}$$

and the projections of the spin vector on X and Z axes are

$$S_1 = D_3^2 \int\limits_{-\infty}^{\infty} \sinh[2(\mathcal{A} + D_4)] \mathrm{e}^{\alpha - \chi} \mathrm{d}x, \tag{3.64a}$$

$$S_3 = D_3^2 \int\limits_{-\infty}^{\infty} \mathrm{e}^{\alpha - \chi} \mathrm{d}x. \tag{3.64b}$$

For α, therefore for P we have

$$\int \frac{\mathrm{d}P}{\mathcal{L}(P)} = \pm(x + x_0), \quad \mathcal{L}(P) = \sqrt{B^2 P^2 - 3\kappa C_0^2 \left(-F(P) + 2\Omega \frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} - \Psi(\Omega) \right)} \tag{3.65}$$

As one sees, Eq. (3.65) coincides with (3.14) if one sets spinor mass $m = 0$ in (3.14). We would also note that for $F = K_\pm$ with $K_\pm = I \pm J$ in case of massless spinor field we obtain $K_\pm = K_0 \mathrm{e}^{-2\alpha}$ and equation for α, hence for $K_\pm$ will be identical to (3.65). A detailed description of spinor field components as well as the metric functions for these cases can be found in [17].

C. Nonlinear scalar field in absence of spinor one

Let us consider the system of gravitational and nonlinear scalar fields. As a nonlinear scalar field equation we choose Born–Infeld one, given by the Lagrangian [12]:

$$\Psi(\Omega) = -\frac{1}{\sigma} \left(1 - \sqrt{1 + \sigma\Omega} \right), \tag{3.66}$$

with $\Omega = \varphi_\alpha \varphi^\alpha$ and σ is the parameter of nonlinearity. Here we would like to note that recently there has been a renewal interest for the Born–Infeld theory due partly to its link with relativistic strings, membranes [18–20] and gravitation theory [21] and partly to its nonlinear structure. From a natural generalization of Born–Infeld Lagrangian, Boillat and Strumia [22] obtained solutions that appear to reflect the influence of the states of spin on the electro-magnetic field.

Let us go back to our initial problem. From (3.66) we also have

$$\lim_{\sigma \to 0} \Psi(\Omega) = \tfrac{1}{2} \Omega \cdots . \tag{3.67}$$

Inserting (3.66) into (2.21) for the scalar field we obtain the equation

$$\varphi'(x) = \frac{\varphi_0}{\sqrt{1 + \sigma \varphi_0^2 \mathrm{e}^{-2\alpha(x)}}}, \tag{3.68}$$

that gives

$$\Omega = -(\varphi')^2 \mathrm{e}^{-2\alpha} = -\frac{\varphi_0^2 \mathrm{e}^{-2\alpha(x)}}{1 + \sigma \varphi_0^2 \mathrm{e}^{-2\alpha(x)}}. \tag{3.69}$$

From (3.68) follows that $\varphi' \,|_{\sigma=0} = \varphi_0\,.$

For the case considered in this section we have

$$T_{\rm sc0}^{\;0} = T_{\rm sc2}^{\;2} = T_{\rm sc3}^{\;3} = -\Psi(\Omega) = \frac{1}{\sigma}\left(1 - \frac{1}{\sqrt{1+\sigma\varphi_0^2 \mathrm{e}^{-2\alpha(x)}}}\right) \tag{3.70}$$

and

$$T_{\rm sc1}^{\;1} = 2\Omega\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} - \Psi = \frac{1}{\sigma}\left(1 - \sqrt{1+\sigma\varphi_0^2 \mathrm{e}^{-2\alpha(x)}}\right). \tag{3.71}$$

Putting (3.71) into (2.44), in account of $m = 0$ and $F(I,J) \equiv 0$ for α we find

$$\alpha' = \pm\sqrt{B^2 - \frac{3\kappa}{\sigma}\mathrm{e}^{2\alpha}\left(1 - \sqrt{1+\sigma\varphi_0^2 \mathrm{e}^{-2\alpha(x)}}\right)}. \tag{3.72}$$

Integrating (3.72) one finds

$$\int \frac{\mathrm{d}\alpha}{\sqrt{B^2 - \frac{3\kappa}{\sigma}\mathrm{e}^{2\alpha}\left(1 - \sqrt{1+\sigma\varphi_0^2 \mathrm{e}^{-2\alpha(x)}}\right)}} = -\frac{2}{B}\ln\left|\xi + \sqrt{\bar\kappa + \xi^2}\right| \tag{3.73}$$

$$+ \frac{1}{B\sqrt{1+\frac{1}{2}\bar\kappa}}\left[\ln\left|\sqrt{2}B\sqrt{\bar\kappa+\xi^2} + \sqrt{2}B\sqrt{1+\tfrac{1}{2}\bar\kappa\xi}\right| - \ln\left|\sqrt{3\kappa\varphi_0^2(\xi^2-2)}\right|\right] = x$$

with $\xi^2 = 1 + \sqrt{1+\sigma\varphi_0^2\mathrm{e}^{-2\alpha(x)}}$. From (3.73) it follows that

$$\mathrm{e}^{2\alpha(x)}\,|_{x\to+\infty} \;\approx\; \tfrac{1}{2}\sigma\varphi_0^2\mathrm{e}^{2\sqrt{1+\bar\kappa/2}Bx} \to \infty, \tag{3.74}$$

$$\mathrm{e}^{2\alpha(x)}\,|_{x\to-\infty} \;\approx\; \tfrac{1}{2}\sigma\varphi_0^2\mathrm{e}^{2Bx} \to 0. \tag{3.75}$$

Let us study the energy density distribution of nonlinear scalar field. From (3.70) we find

$$T_{\rm sc0}^{\;0}(x)\,|_{x=-\infty} = \frac{1}{\sigma}, \quad T_{\rm sc0}^{\;0}(x)\,|_{x=\infty} = 0, \tag{3.76}$$

which shows that the energy density of the scalar field is not localized. Nevertheless, the energy density on unit invariant volume is localized if $\kappa\varphi_0^2 > 2B^2$:

$$\varepsilon(x) = T_{\rm sc0}^{\;0}\sqrt{-^3g} = \frac{1}{\sigma}\left(1 - \frac{1}{1+\sigma\varphi_0^2\mathrm{e}^{-2\alpha}}\right)\mathrm{e}^{5\alpha/3+2Bx/3}\Bigg|_{x\to\pm\infty} \to 0. \tag{3.77}$$

In this case the total energy of the scalar field is also bound. From (3.69) in account of (3.74) and (3.75) we also have

$$\Omega(x)\,|_{x=-\infty} = -\frac{1}{\sigma}, \quad \Omega(x)\,|_{x=+\infty} = 0, \tag{3.78}$$

showing that $\Omega(x)$ is kink-like.

D. Nonlinear spinor and nonlinear scalar field

Finally we consider the self-consistent system of nonlinear spinor and scalar fields. We choose the self-action of the spinor field as $F = \lambda S^n$, $n > 2$, where as the scalar field is taken in the form (3.66). Using the line of reasoning mentioned earlier, we conclude that the spinor field considered here should be massless. Taking into account that $\mathrm{e}^{-2\alpha} = S^2/C_0^2$ for S, we write

$$\int \frac{\mathrm{d}S}{\sqrt{B^2 + 3\kappa C_0^2 \left[\lambda S^n + \left(\sqrt{1+\sigma\varphi_0^2 \frac{S^2}{C_0^2}} - 1\right)\frac{1}{\sigma}\right]}} = x. \tag{3.79}$$

From (3.79) one estimates

$$S(x)\,|_{x\to 0} \approx \frac{1}{x^{2/(n-2)}} \to \infty. \tag{3.80}$$

On the other hand for the energy density we have

$$T_0^0 = \lambda(n-1)S^n + \frac{1}{\sigma}\left(1 - \frac{1}{\sqrt{1+\sigma\varphi_0^2\frac{S^2}{C_0^2}}}\right) \tag{3.81}$$

that states that for T_0^0 to be localized S should be localized too and $\lim\limits_{x\to\pm\infty} S(x) \to 0$. Hence from (3.80) we conclude that $S(x)$ is singular and energy density in unlimited at $x = 0$.

For $\lambda = -\Lambda^2$ and $n > 2$ we have

$$\int \frac{\mathrm{d}S}{\sqrt{B^2 + 3\kappa C_0^2 \left[-\Lambda^2 S^n + \left(\sqrt{1+\sigma\varphi_0^2 \frac{S^2}{C_0^2}} - 1\right)\frac{1}{\sigma}\right]}} = x. \tag{3.82}$$

In this case $S(x)$ is finite and its maximum value is defined from

$$S^n(x) = \frac{1}{3\kappa C_0^2 \Lambda^2}\left[B^2 S^2 + 3\kappa C_0^2 \left(\sqrt{1+\sigma\varphi_0^2\frac{S^2}{C_0^2}} - 1\right)\frac{1}{\sigma}\right]. \tag{3.83}$$

Noticing that at spatial infinity effects of nonlinearity vanish, from (3.82) we find

$$S(x)\,\big|_{x\to-\infty} \approx \mathrm{e}^{Hx} \to 0\,, \quad S(x)\,\big|_{x\to+\infty} \approx \mathrm{e}^{-Hx} \to 0\,, \tag{3.84}$$

with $H = \sqrt{B^2 + 3\kappa\varphi_0^2/2} = B\sqrt{1+\bar{\kappa}/2}$. In this case the energy density T_0^0 defined by (3.81) is localized and the total energy of the system in bound. Nevertheless, spin and charge of the system are unlimited.

Let us go back to the general case. For $F = F(S)$ we now have

$$T_1^1 = mS - F(S) + 2\Omega \frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} - \Psi. \tag{3.85}$$

It follows that for the arbitrary choice of $\Psi(\Omega)$, obeying (2.4), we can always choose nonlinear spinor term that will eliminate the scalar field contribution in T_1^1, i.e., by virtue of total freedom we have here to choose $F(S)$, we can write

$$F(S) = F_1(S) + F_2(S), \quad F_2(S) = 2\Omega \frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} - \Psi, \tag{3.86}$$

since $\Omega = \Omega(S^2)$. To prove this we go back to (2.21) that gives

$$\Omega \left(\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega}\right)^2 = -\frac{\varphi_0^2 S^2}{C_0^2}. \tag{3.87}$$

Since Ψ is the function of Ω only, (3.87) comprises an algebraic equation for defining Ω as a function of S^2. For (3.86) takes place, we find

$$(\alpha')^2 - B^2 = -\frac{3\kappa C_0^2}{S^2}\left[mS - F_1(S)\right]. \tag{3.88}$$

As we see, the scalar field has no effect on space-time, but it contributes to energy density and total energy of the system as in this case:

$$T_0^0 = SF_1'(S) - F_1(S) + S\frac{\mathrm{d}}{\mathrm{d}\Omega}\left(-2\Omega\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} + \Psi\right)\frac{\mathrm{d}\Omega}{\mathrm{d}S} + 2\Omega\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} - \Psi. \tag{3.89}$$

Note that in (3.85) with $F(S)$ arbitrary, we cannot choose $\Psi(\Omega)$ such that

$$2\Omega\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} - \Psi = F(S), \tag{3.90}$$

due to the fact that $\Psi(\Omega)$ is not totally arbitrary, since it has to obey

$$\lim_{\Omega\to 0}\Psi(\Omega) \to \frac{1}{2}\Omega, \quad \lim_{\Omega\to 0} 2\Omega\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} - \Psi = \frac{1}{2}\Omega = \frac{\varphi_0^2}{2C_0^2}S^2\,, \tag{3.91}$$

whereas at $S \to 0$, $F(S)$ behaves arbitrarily.

4 Conclusion and discussion

The system of nonlinear spinor and nonlinear scalar fields with minimal coupling has been thoroughly studied within the scope of general relativity given by a plane-symmetric space-time. Contrary to the scalar field, the spinor field nonlinearity has direct effect on space-time. Energy density and the total energy of the linear spinor and scalar field system are not bounded and the system does not possess real physical infinity, hence the configuration is not observable for an infinitely remote observer, since in this case

$$R = \int_{-\infty}^{\infty} \sqrt{g_{11}}\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{\alpha}\mathrm{d}x = \frac{4C_0H}{M^2} < \infty. \tag{4.1}$$

But introduction of nonlinear spinor term into the system eliminates these shortcomings and we have the configuration with finite energy density and limited total energy which is also observable as in this case the system possesses real physical infinity. Thus we see, spinor field nonlinearity is crucial for the regular solutions with localized energy density. We also conclude that the properties of nonlinear spinor and scalar field system with minimal coupling are defined by that part of gravitational field which is generated by nonlinear spinor one. We would also like to note that though the title of the paper may indicate otherwise, this report is in no way devoted to finding and interpretations of soliton-like solutions to the nonlinear field equations let alone the investigation of the possibilities of existence of multidimensional solitons. As it was mentioned in the introduction and showed later on, the gravitational field plays crucial role in the formation of soliton-like solutions (solutions obeying three requirements formulated is Sect. 2) and it is the central issue of the present report.

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