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Interacting Spinor and Scalar Fields in Bianchi Type I Universe Filled with Perfect Fluid: Exact Self-consistent Solutions

B. Saha[1] **and G. N. Shikin**[2]

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In the framework of Bianchi I (BI) cosmological models a self-consistent system of interacting spinor and scalar fields has been considered. We introduced an interaction function $F(I, J)$ which is an arbitrary function of invariants I and J, generated from the real bilinear forms of the spinor field. Exact self-consistent solutions to the field equations have been obtained for the cosmological model filled with perfect fluid. The initial and the asymptotic behavior of the field functions and of the metric one has been thoroughly studied.

KEY WORDS : Cosmological model with spinor and scalar field

1. INTRODUCTION

The quantum field theory in curved space-time has been a matter of great interest in recent years because of its applications to cosmology and astrophysics. The evidence of existence of strong gravitational fields in our universe led to the study of quantum effects of matter fields in an external classical gravitational field. After the appearance of Parker's paper on scalar fields [1] and spin-$\frac{1}{2}$ fields [2], several authors have studied this subject. Although the universe seems homogenous and isotropic at present,

[1] Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, 141980 Dubna, Moscow region, Russia. E-mail: saha@thsun1.jinr.dubna.su
[2] Department of Theoretical Physics, Russian Peoples' Friendship University, 6, Miklukho-Maklay str., 117198 Moscow, Russia

there is no observational data that guaranties the isotropy in the era prior to the recombination. In fact, there are theoretical arguments that sustain the existence of an anisotropic phase that approaches an isotropic one [3]. Interest in studying Klein–Gordon and Dirac equations in anisotropic models has increased since Hu and Parker [4] have shown that the creation of scalar particles in anisotropic backgrounds can dissipate the anisotropy as the universe expands.

A Bianchi type I (BI) universe, being the straightforward generalization of the flat Robertson–Walker (RW) universe, is one of the simplest models of an anisotropic universe that describes a homogenous and spatially flat universe. Unlike the RW universe which has the same scale factor for each of the three spatial directions, a BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. It moreover has the agreeable property that near the singularity it behaves like a Kasner universe even in the presence of matter and consequently falls within the general analysis of the singularity given by Belinskii et al. [5]. And in a universe filled with matter for $p = \gamma \varepsilon$, $\gamma < 1$, it has been shown that any initial anisotropy in a BI universe quickly dies away and a BI universe eventually evolves into a RW universe [6]. Since the present-day universe is surprisingly isotropic, this feature of the BI universe makes it a prime candidate for studying the possible effects of an anisotropy in the early universe on present-day observations. In light of the importance of what has been mentioned above, several authors have studied linear spinor field equations [7,8] and the behavior of gravitational waves (GWS) [9–11] in the BI universe. A nonlinear spinor field (NLSF) in an external cosmological gravitational field was first studied by G. N. Shikin in 1991 [12]. This study was extended by us to the more general case where we consider the nonlinear term as an arbitrary function of all possible invariants generated from spinor bilinear forms. In that paper we also studied the possibility of elimination of initial singularity especially for a Kasner universe [13]. In a recent paper [14] we studied the behavior of self-consistent NLSF in the BI universe, and that was followed by [15,16], where we studied the self-consistent system of interacting spinor and scalar fields.

The purpose of the present paper is to extend our study on different kinds of interacting term in presence of perfect fluid. Earlier we considered the function $F(I)$ [15,16] that describes the interaction between spinor fields and scalar ones, which is an arbitrary function of invariant $I = S^2$, where $S = \bar{\psi}\psi$, generated from the real bilinear forms of the spinor field. Herein we introduce interaction function $F(I, J)$ containing an additional argument $J = P^2$ with $P = i\bar{\psi}\gamma^5\psi$. Contrary to the previous papers [15,16] the cosmological model in this case contains a perfect fluid. Moreover, here we consider three types of interactions between the spinor and scalar fields. In Section 2 we derive fundamental equations corresponding to the Lagrangian for the self-consistent system of spinor, scalar and gravitational fields in presence of perfect fluid and seek their general solutions. In Section 3 we give a detailed analysis of the solutions obtained for different kinds of interacting term. In Section 4 we sum up the results obtained.

2. FUNDAMENTAL EQUATIONS AND GENERAL SOLUTIONS

The Lagrangian for the self-consistent system of spinor, scalar and gravitational fields in the presence of a perfect fluid is

$$L = L_{\rm g} + L_{\rm sp} + L_{\rm sc} + L_m + L_{\rm int}, \tag{1}$$

where $L_{\rm g}$, $L_{\rm sp}$, $L_{\rm sc}$, correspond to gravitational, free spinor and free scalar fields read

$$\begin{aligned} L_{\rm g} &= R/2\kappa, \\ L_{\rm sp} &= (i/2)\,[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi] - m\bar\psi\psi, \\ L_{\rm sc} &= \tfrac{1}{2}\varphi_{,\mu}\varphi^{,\mu}, \end{aligned}$$

with R being the scalar curvature, κ the Einstein's gravitational constant and L_m the Lagrangian of the perfect fluid. As interaction Lagrangian we consider the following cases [15–17]:

$$\begin{aligned} &\text{(i)} & L_{\rm int} &= (\lambda/2)\,\varphi_{,\alpha}\varphi^{,\alpha}F, \\ &\text{(ii)} & L_{\rm int} &= \lambda\bar\psi\gamma^\mu\psi\varphi_{,\mu}, \\ &\text{(iii)} & L_{\rm int} &= i\lambda\bar\psi\gamma^\mu\gamma^5\psi\varphi_{,\mu}, \end{aligned}$$

where λ is the coupling constant and F can be presented as some arbitrary functions of invariants generated from the real bilinear forms of spinor field having the form

$$\begin{aligned} S &= \bar\psi\psi, \qquad P = i\bar\psi\gamma^5\psi, \qquad v^\mu = (\bar\psi\gamma^\mu\psi), \\ A^\mu &= (\bar\psi\gamma^5\gamma^\mu\psi), \qquad T^{\mu\upsilon} = (\bar\psi\sigma^{\mu\upsilon}\psi), \end{aligned}$$

where $\sigma^{\mu\upsilon} = (i/2)[\gamma^\mu\gamma^\upsilon - \gamma^\upsilon\gamma^\mu]$. Invariants, corresponding to the bilinear forms, appear as

$$\begin{aligned} I = S^2, \qquad J = P^2, \qquad I_A &= A_\mu A^\mu = (\bar\psi\gamma^5\gamma^\mu\psi)\,g_{\mu\upsilon}(\bar\psi\gamma^5\gamma^\upsilon\psi), \\ I_v &= v_\mu v^\mu = (\bar\psi\gamma^\mu\psi)\,g_{\mu\upsilon}(\bar\psi\gamma^\upsilon\psi), \\ I_T &= T_{\mu\upsilon}T^{\mu\upsilon} = (\bar\psi\sigma^{\mu\upsilon}\psi)\,g_{\mu\alpha}g_{\upsilon\beta}(\bar\psi\sigma^{\alpha\beta}\psi). \end{aligned}$$

According to the Pauli–Fierz theorem [18] among the five invariants only I and J are independent, as all other can be expressed by them:

$$I_v = -I_A = I + J, \qquad I_T = I - J.$$

Therefore we choose $F = F(I, J)$.

We choose the BI space-time metric in the form

$$ds^2 = dt^2 - \gamma_{ij}(t)\, dx^i\, dx^j\,. \tag{2}$$

As it admits no rotational matter, the spatial metric $\gamma_{ij}(t)$ can be put into diagonal form. Now we can rewrite the BI space-time metric in the form [19]

$$ds^2 = dt^2 - a^2(t)\, dx^2 - b^2(t)\, dy^2 - c^2(t)\, dz^2, \tag{3}$$

where the velocity of light c is taken to be unity.

Let us now write the Einstein equations for $a(t)$, $b(t)$ and $c(t)$ corresponding to the metric (3) and Lagrangian (1) [19]:

$$\frac{\ddot a}{a} + \frac{\dot a}{a}\left(\frac{\dot b}{b} + \frac{\dot c}{c}\right) = -\kappa\left(T_1^1 - \frac{1}{2}T\right), \tag{4}$$

$$\frac{\ddot b}{b} + \frac{\dot b}{b}\left(\frac{\dot a}{a} + \frac{\dot c}{c}\right) = -\kappa\left(T_2^2 - \frac{1}{2}T\right), \tag{5}$$

$$\frac{\ddot c}{c} + \frac{\dot c}{c}\left(\frac{\dot a}{a} + \frac{\dot b}{b}\right) = -\kappa\left(T_3^3 - \frac{1}{2}T\right), \tag{6}$$

$$\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c} = -\kappa\left(T_0^0 - \frac{1}{2}T\right), \tag{7}$$

where points denote differentiation with respect to t, and T_μ^υ is the energy-momentum tensor of material fields and perfect fluid.

The scalar and the spinor field equations and the energy-momentum tensor of material fields and perfect fluid corresponding to (1) are

$$\partial_\alpha\left[\sqrt{-g}\left(g^{\alpha\beta}\varphi_{,\beta} + \partial L_{\text{int}}/\partial\varphi_{,\alpha}\right)\right] = 0, \tag{8}$$

$$\begin{aligned} i\gamma^\mu\nabla_\mu\psi - m\psi + \partial L_{\text{int}}/\partial\bar\psi &= 0,\\ i\nabla_\mu\bar\psi\gamma^\mu + m\bar\psi - \partial L_{\text{int}}/\partial\psi &= 0\,. \end{aligned} \tag{9}$$

$$\begin{aligned} T_\mu^\rho = {} & \frac{i}{4}g^{\rho\upsilon}\left(\bar\psi\gamma_\mu\nabla_\upsilon\psi + \bar\psi\gamma_\upsilon\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\upsilon\psi - \nabla_\upsilon\bar\psi\gamma_\mu\psi\right)\\ & + \varphi_{,\mu}\varphi^{,\rho} + 2\frac{\delta L_{\text{int}}}{\delta g^{\mu\upsilon}}g^{\rho\upsilon} - \delta_\mu^\rho\left(L_{\text{sp}} + L_{\text{sc}} + L_{\text{int}}\right) + T_{\mu\,(m)}^\rho. \end{aligned} \tag{10}$$

Here $T^{\rho}_{\mu\,(m)}$ is the energy-momentum tensor of perfect fluid. For a universe filled with perfect fluid, in the concomitant system of reference ($u^0 = 1$, $u^i = 0$, $i = 1, 2, 3$) we have

$$T^{\upsilon}_{\mu(m)} = (p + \varepsilon)u_{\mu}u^{\upsilon} - \delta^{\upsilon}_{\mu}p = (\varepsilon, -p, -p, -p), \tag{11}$$

where energy ε is related to the pressure p by the equation of state $p = \gamma\,\varepsilon$, the general solution has been derived by Jacobs [6]. γ varies between the interval $0 \leq \gamma \leq 1$; whereas $\gamma = 0$ describes the dust universe, $\gamma = \frac{1}{3}$ presents the radiation universe, $\frac{1}{3} < \gamma < 1$ the hard universe and $\gamma = 1$ corresponds to the stiff matter. One sees changes in the solutions performed by perfect fluid carried out through Einstein equations, namely through $\tau = a(t)b(t)c(t)$. Note that a perfect fluid appears in the system through the energy-momentum tensor. In the field equations it does not appear directly but acts on the fields through the metric functions. With the perfect fluid taken into account, one can find metric functions solving the Einstein equations. So, we first see how the quantities ε and p are connected with the metric functions, namely with τ where $\tau := \sqrt{-g} = abc$. In doing this we use the well-known equality $T^{\upsilon}_{\mu;\upsilon} = 0$, which leads to

$$\frac{d}{dt}(\tau\varepsilon) + \dot{\tau}p = 0, \tag{12}$$

with the solution

$$\ln\tau = -\int\frac{d\varepsilon}{(\varepsilon + p)}. \tag{13}$$

Recalling the equation of state $p = \xi\varepsilon$, $0 \leq \xi \leq 1$ finally we get

$$\begin{aligned} T^{0}_{0(m)} &= \varepsilon = \frac{\varepsilon_0}{\tau^{1+\xi}}, \\ T^{1}_{1(m)} &= T^{2}_{2(m)} = T^{3}_{3(m)} = -p = -\frac{\varepsilon_0\xi}{\tau^{1+\xi}}, \end{aligned} \tag{14}$$

where ε_0 is the integration constant.

Note that we consider space-independent fields only. Under this assumption and with regard to spinor field equations, the components of the energy-momentum tensor read:

$$\begin{aligned} T^{0}_{0} &= mS + \frac{1}{2}\dot{\varphi}^2 + L_{\rm int} + \varepsilon, \\ T^{1}_{1} &= T^{2}_{2} = T^{3}_{3} = \frac{1}{2}\left(\bar{\psi}\frac{\partial L_{\rm int}}{\partial\bar{\psi}} + \frac{\partial L_{\rm int}}{\partial\psi}\psi\right) - L_{\rm sc} - L_{\rm int} - p. \end{aligned} \tag{15}$$

In (8) and (10) ∇_μ denotes the covariant derivative of spinor, having the form [20]

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \tag{16}$$

where $\Gamma_\mu(x)$ are spinor affine connection matrices. $\gamma^\mu(x)$ matrices are defined for the metric (3) as follows. Using the equalities [21,22]

$$g_{\mu\upsilon}(x) = e_\mu^a(x) e_\upsilon^b(x) \eta_{ab}, \qquad \gamma_\mu(x) = e_\mu^a(x) \bar{\gamma}^a,$$

where $\eta_{ab} = \mathrm{diag}\,(1, -1, -1, -1)$, $\bar{\gamma}_\alpha$ are the Dirac matrices of Minkowski space and $e_\mu^a(x)$ are the set of tetradic 4-vectors, we obtain the Dirac matrices $\gamma^\mu(x)$ of curved space-time,

$$\gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \bar{\gamma}^1/a(t), \quad \gamma^2 = \bar{\gamma}^2/b(t), \quad \gamma^3 = \bar{\gamma}^3/c(t),$$
$$\gamma_0 = \bar{\gamma}_0, \quad \gamma_1 = \bar{\gamma}_1 a(t), \quad \gamma_2 = \bar{\gamma}_2 b(t), \quad \gamma_3 = \bar{\gamma}_3 c(t).$$

$\Gamma_\mu(x)$ matrices are defined by the equality

$$\Gamma_\mu(x) = \tfrac{1}{4} g_{\rho\sigma}(x) \left(\partial_\mu e_\delta^b e_b^\rho - \Gamma_{\mu\delta}^\rho\right) \gamma^\sigma \gamma^\delta,$$

which gives

$$\begin{aligned} \Gamma_0 &= 0, & \Gamma_1 &= \tfrac{1}{2}\dot{a}(t)\bar{\gamma}^1\bar{\gamma}^0, \\ \Gamma_2 &= \tfrac{1}{2}\dot{b}(t)\bar{\gamma}^2\bar{\gamma}^0, & \Gamma_3 &= \tfrac{1}{2}\dot{c}(t)\bar{\gamma}^3\bar{\gamma}^0. \end{aligned} \tag{17}$$

Flat space-time matrices we choose in the form, given in [23],

$$\bar{\gamma}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \bar{\gamma}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\gamma}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \bar{\gamma}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Defining γ^5 as follows:

$$\gamma^5 = -\frac{i}{4} E_{\mu\upsilon\sigma\rho} \gamma^\mu \gamma^\upsilon \gamma^\sigma \gamma^\rho, \qquad E_{\mu\upsilon\sigma\rho} = \sqrt{-g}\,\varepsilon_{\mu\upsilon\sigma\rho}, \qquad \varepsilon_{0123} = 1,$$

$$\gamma^5 = -i\sqrt{-g}\,\gamma^0\gamma^1\gamma^2\gamma^3 = -i\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3 = \bar{\gamma}^5,$$

we obtain

$$\bar{\gamma}^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let us now solve the Einstein equations. With respect to (15) summation of Einstein equations (4), (5) and (6) leads to the equation

$$\frac{\ddot{\tau}}{\tau} = -\kappa\left(T_1^1 + T_2^2 + T_3^3 - \frac{3}{2}T\right) = \frac{3\kappa}{2}(T_0^0 + T_1^1). \tag{18}$$

If the right hand side of (18) is the function of $\tau(t) = a(t)b(t)c(t)$, this equation takes the form

$$\ddot{\tau} + \Phi(\tau) = 0, \tag{19}$$

which possesses exact solutions for arbitrary function $\Phi(\tau)$. Note that the assumption of the right hand side of (18) to be the function of τ is not a general one, though it is always a function of t for the space-independent field functions. But as will be shown later, in the particular case considered here, the r.h.s. of (18) is a function of τ. Given the explicit form of $L_{\rm int}$, from (18) one can find concrete function $\tau(t) = abc$. Once the value of τ is obtained, one can get expressions for components $V_\alpha(t)$, $\alpha = 1, 2, 3, 4$. Let us express a, b, c through τ. For this we notice that subtraction of Einstein equations (4)–(5) leads to the equation

$$\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{c}}{ac} - \frac{\dot{b}\dot{c}}{bc} = \frac{d}{dt}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) + \left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) = 0. \tag{20}$$

Equation (20) possesses the solution

$$\frac{a}{b} = D_1 \exp\left(X_1 \int \frac{dt}{\tau}\right), \qquad D_1 = \text{const.}, \quad X_1 = \text{const.} \tag{21}$$

Subtracting eqs. (4)–(6) and (5)–(6) one finds the equations similar to (20), having solutions

$$\frac{a}{c} = D_2 \exp\left(X_2 \int \frac{dt}{\tau}\right), \qquad \frac{b}{c} = D_3 \exp\left(X_3 \int \frac{dt}{\tau}\right), \tag{22}$$

where D_2, D_3, X_2, X_3 are integration constants. There is a functional dependence between the constants D_1, D_2, D_3, X_1, X_2, X_3:

$$D_2 = D_1 D_3, \qquad X_2 = X_1 + X_3.$$

Using the equations (21) and (22), we rewrite $a(t)$, $b(t)$, $c(t)$ in the explicit form:

$$a(t) = (D_1^2 D_3)^{1/3} \tau^{1/3} \exp\left[\frac{2X_1 + X_3}{3} \int \frac{dt}{\tau(t)}\right],$$
$$b(t) = (D_1^{-1} D_3)^{1/3} \tau^{1/3} \exp\left[-\frac{X_1 - X_3}{3} \int \frac{dt}{\tau(t)}\right],$$
$$c(t) = (D_1 D_3^2)^{-1/3} \tau^{1/3} \exp\left[-\frac{X_1 + 2X_3}{3} \int \frac{dt}{\tau(t)}\right]. \qquad (23)$$

Thus the previous system of Einstein equations is completely integrated. In this process of integration only the first three of the complete system of Einstein equations have been used. General solutions to these three second order equations have been obtained. The solutions contain six arbitrary constants: D_1, D_3, X_1, X_3 and two others, that were obtained while solving eq. (19). Equation (7) is the consequence of the first three Einstein equations. To verify the correctness of the obtained solutions, it is necessary to put a, b, c into (7). It should lead either to identity or to some additional constraint between the constants. Putting a, b, c from (23) into (7) one can get the following equality:

$$\frac{\ddot{\tau}}{\tau} - \frac{2}{3}\frac{\dot{\tau}^2}{\tau^2} + \frac{2}{9\tau^2} X = -\frac{\kappa}{2}(T_0^0 - 3T_1^1), \quad X := X_1^2 + X_1 X_3 + X_3^2, \qquad (24)$$

that guarantees the correctness of the solutions obtained.

It should be emphasized that we are dealing with a cosmological problem and our main goal is to investigate the initial and the asymptotic behavior of the field functions and the metric functions. As one sees, all these functions are in some functional dependence on τ. Therefore in our further investigation we mainly look for τ, though in some particular cases we write down field and metric functions explicitly.

3. ANALYSIS OF THE SOLUTIONS OBTAINED FOR SOME SPECIAL CHOICE OF INTERACTION LAGRANGIAN

Let us now study the system for some special choice of $L_{\rm int}$. We first study the solution to the system of field equations with minimal coupling when the direct interaction between the spinor and scalar fields remains absent. The reason for obtaining the solution to the self-consistent system of equations for the fields with minimal coupling is the need to compare this solution with that for the system of equations for the interacting spinor, scalar and gravitational fields that permits us to clarify the role of interaction terms in the evolution of the cosmological model in question.

In this case from the scalar and spinor field equations one finds $\dot{\varphi} = C/\tau$ and $\bar{\psi}\psi = S = C_0/\tau$ with C and C_0 being the constants of integration. Therefore the components of the energy-momentum tensor appear as

$$T_0^0 = \frac{m C_0}{\tau} + \frac{C^2}{2\tau^2}, \qquad T_1^1 = T_2^2 = T_3^3 = -\frac{C^2}{2\tau^2}. \tag{25}$$

Since in (25) T_0^0 is the energy density of free fields, we assume that T_0^0 is positively defined. This leads to $C_0 > 0$. The inequality $C_0 > 0$ will also be preserved for the system with direct interaction between the fields, as in this case the correspondence principle should be fulfilled: for $\lambda = 0$ the field system with direct interaction turns into that with minimal coupling.

The components of spinor field functions in this case read

$$\psi_{1,2}(t) = (C_{1,2}/\sqrt{\tau})\, e^{-imt}, \qquad \psi_{3,4}(t) = (C_{3,4}/\sqrt{\tau})\, e^{imt}. \tag{26}$$

Taking into account (25), from (18) and (24) one gets

$$\tau|_{t\to 0} \approx \sqrt{3\kappa C^2/2 + X/3}\, t \to 0, \qquad \tau|_{t\to\infty} \approx \sqrt{3\kappa m C_0}\, t^2.$$

Thus one concludes that the solutions obtained are initially singular and the space-time is asymptotically isotropic.

Let us now study the case with different kinds of interactions.

Case 1. For the case when $L_{\rm int} = (\lambda/2)\varphi_{,\mu}\varphi^{,\mu}F(I,J)$ one writes the scalar field equation as

$$\frac{\partial}{\partial t}(\tau\dot{\varphi}(1 + \lambda F)) = 0, \tag{27}$$

with the solution

$$\dot{\varphi} = C/\tau(1 + \lambda F). \tag{28}$$

In this case the first equation of the system (9) now reads

$$i\bar{\gamma}^0\left(\frac{\partial}{\partial t} + \frac{\dot{\tau}}{2\tau}\right)\psi - m\psi + \mathcal{D}\psi + i\mathcal{G}\gamma^5\psi = 0, \tag{29}$$

where $\mathcal{D} := \varphi_{,\alpha}\varphi^{,\alpha} S F_I$ and $\mathcal{G} := \varphi_{,\alpha}\varphi^{,\alpha} P F_J$. For the components $\psi_\rho = V_\rho(t)$, where $\rho = 1, 2, 3, 4,$ from (29) one deduces the following system of equations:

$$\dot{V}_1 + \frac{\dot{\tau}}{2\tau} V_1 + i(m - \mathcal{D}) V_1 - \mathcal{G} V_3 = 0,$$
$$\dot{V}_2 + \frac{\dot{\tau}}{2\tau} V_2 + i(m - \mathcal{D}) V_2 - \mathcal{G} V_4 = 0,$$
$$\dot{V}_3 + \frac{\dot{\tau}}{2\tau} V_3 - i(m - \mathcal{D}) V_3 + \mathcal{G} V_1 = 0,$$
$$\dot{V}_4 + \frac{\dot{\tau}}{2\tau} V_4 - i(m - \mathcal{D}) V_4 + \mathcal{G} V_2 = 0. \tag{30}$$

Let us now define the equations for

$$P = i(V_1 V_3^* - V_1^* V_3 + V_2 V_4^* - V_2^* V_4),$$
$$R = (V_1 V_3^* + V_1^* V_3 + V_2 V_4^* + V_2^* V_4),$$
$$S = (V_1^* V_1 + V_2^* V_2 - V_3^* V_3 - V_4^* V_4). \tag{31}$$

After a little manipulation one finds

$$\frac{dS_0}{dt} - 2\mathcal{G} R_0 = 0,$$
$$\frac{dR_0}{dt} + 2(m - \mathcal{D}) P_0 + 2\mathcal{G} S_0 = 0,$$
$$\frac{dP_0}{dt} - 2(m - \mathcal{D}) R_0 = 0, \tag{32}$$

where $S_0 = \tau S$, $P_0 = \tau P$, $R_0 = \tau R$. From this system one can easily find

$$S_0 \dot{S}_0 + R_0 \dot{R}_0 + P_0 \dot{P}_0 = 0,$$

which gives

$$S^2 + R^2 + P^2 = A^2/\tau^2, \qquad A^2 = \text{const}. \tag{33}$$

Let us go back to the system of equations (30). It can be written as follows if one defines $W_\alpha = \sqrt{\tau}\, V_\alpha$:

$$\dot{W}_1 + i\Phi W_1 - \mathcal{G} W_3 = 0, \qquad \dot{W}_2 + i\Phi W_2 - \mathcal{G} W_4 = 0,$$
$$\dot{W}_3 - i\Phi W_3 + \mathcal{G} W_1 = 0, \qquad \dot{W}_4 - i\Phi W_4 + \mathcal{G} W_2 = 0, \tag{34}$$

where $\Phi = m - \mathcal{D}$. Defining $U(\sigma) = W(t)$, where $\sigma = \int \mathcal{G} dt$, we rewrite the foregoing system as

$$U_1' + i(\Phi/\mathcal{G}) U_1 - U_3 = 0, \qquad U_2' + i(\Phi/\mathcal{G}) U_2 - U_4 = 0,$$
$$U_3' - i(\Phi/\mathcal{G}) U_3 + U_1 = 0, \qquad U_4' - i(\Phi/\mathcal{G}) U_4 + U_2 = 0, \tag{35}$$

where prime ($'$) denotes differentiation with respect to σ. One can now define V_α giving the explicit value of $L_{\rm int}$.

Let us consider the case when $F = I^n = S^{2n}$. It is clear that in this case $\mathcal{G} = 0$. From (32) we find

$$S = C_0/\tau, \qquad C_0 = \text{const}. \tag{36}$$

As in the case considered F depends only on S, from (36) it follows that $\mathcal{D}$ is a functions of $\tau = abc$. Taking this fact into account, integration of the system of equations (34) leads to the expressions

$$V_r(t) = (C_r/\sqrt{\tau})\, e^{-i\Omega}, \quad r = 1, 2, \quad V_l(t) = (C_l/\sqrt{\tau})\, e^{i\Omega}, \quad l = 3, 4, \tag{37}$$

where C_r and C_l are integration constants and $\Omega = \int \Phi(t) dt$. Putting this solution into (31) one gets

$$S = (C_1^2 + C_2^2 - C_3^2 - C_4^2)/\tau. \tag{38}$$

Comparing it with (36) we find $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$. In this case from (18) and (24) we find

$$\tau(t)\,|_{t\to\infty} \approx \tfrac{3}{4}\kappa m C_0 t^2 \to \infty, \qquad \tau(t)\,|_{t\to 0} \approx \sqrt{\mathcal{X}/3}\, t \to 0.$$

Thus in the case considered, the asymptotical isotropization of the expansion process of initially anisotropic Bianchi type I space-time takes place without the influence of scalar field. For a detail analysis of this case see [15].

We study the system when $F = J^n = P^{2n}$, which means in the case considered $\mathcal{D} = 0$. Unfortunately, we have not yet been able to find the exact solution for the case in question when $m \neq 0$. Therefore we consider the particular case with $m = 0$. Then from (32) one gets

$$P(t) = D_0/\tau, \qquad D_0 = \text{const}. \tag{39}$$

The system of equations (35) in this case reads

$$\begin{aligned} U_1' - U_3 &= 0, \qquad & U_2' - U_4 &= 0, \\ U_3' + U_1 &= 0, \qquad & U_4' + U_2 &= 0. \end{aligned} \tag{40}$$

Differentiating the first equation of system (40) and taking into account the third one we get

$$U_1'' + U_1 = 0, \tag{41}$$

which leads to the solution

$$U_1 = D_1 e^{i\sigma} + iD_3 e^{-i\sigma}, \qquad U_3 = iD_1 e^{i\sigma} + D_3 e^{-i\sigma}. \tag{42}$$

Analogically for U_2 and U_4 one gets

$$U_2 = D_2 e^{i\sigma} + iD_4 e^{-i\sigma}, \qquad U_4 = iD_2 e^{i\sigma} + D_4 e^{-i\sigma}, \tag{43}$$

where D_i are the constants of integration. Finally, we can write

$$\begin{aligned} V_1 &= (1/\sqrt{\tau})(D_1 e^{i\sigma} + iD_3 e^{-i\sigma}), \quad V_2 = (1/\sqrt{\tau})(D_2 e^{i\sigma} + iD_4 e^{-i\sigma}), \\ V_3 &= (1/\sqrt{\tau})(iD_1 e^{i\sigma} + D_3 e^{-i\sigma}), \quad V_4 = (1/\sqrt{\tau})(iD_2 e^{i\sigma} + D_4 e^{-i\sigma}). \end{aligned} \tag{44}$$

Putting (44) into (31) one finds

$$P = 2(D_1^2 + D_2^2 - D_3^2 - D_4^2)/\tau. \tag{45}$$

Comparison of (39) with (45) gives $D_0 = 2(D_1^2 + D_2^2 - D_3^2 - D_4^2)$. Let us now estimate τ. From (18) and (24) we obtain

$$\tau|_{t\to\infty} \approx ([\sqrt{\varepsilon_0}(\xi + 1)/2]\, t)^{2/(\xi+1)}, \qquad \tau|_{t\to 0} \approx \sqrt{\mathcal{X}/3}\, t,$$

i.e. the solutions obtained are initially singular and the space-time is asymptotically isotropic if $\xi < 1$ and anisotropic if $\xi = 1$.

Let us now study the case when $F = F(I, J)$. Choosing

$$F = F(K_\pm), \quad K_+ = I + J = I_v = -I_A, \qquad K_- = I - J = I_T, \tag{46}$$

in case of massless spinor field ($m = 0$) we find

$$\mathcal{D} = \varphi_{,\mu}\varphi^{,\mu} S F_{K_\pm}, \qquad \mathcal{G} = \pm\varphi_{,\mu}\varphi^{,\mu} S F_{K_\pm}, \qquad F_{K_\pm} = dF/dK_\pm.$$

Putting these into (32) we find

$$S_0^2 \pm P_0^2 = D_\pm. \tag{47}$$

Choosing $F = K_\pm^n$ from (18) and (24) one comes to the conclusion similar to that of previous case [$F = F(J)$].

Case 2. In this case the scalar and spinor field equations read

$$\frac{\partial}{\partial t}[\tau(\dot{\varphi} + \lambda\bar{\psi}\gamma^0\psi)] = 0, \tag{48}$$

$$
\begin{aligned}
&i\gamma^0\left(\frac{\partial}{\partial t}+\frac{\dot\tau}{2\tau}\right)\psi - m\psi + \lambda\dot\varphi\gamma^0\psi = 0,\\
&i\left(\frac{\partial}{\partial t}+\frac{\dot\tau}{2\tau}\right)\bar\psi\gamma^0 + m\bar\psi - \lambda\dot\varphi\bar\psi\gamma^0 = 0.
\end{aligned}
\tag{49}
$$

Using the spinor field equations one finds $\bar\psi\gamma^0\psi = C_1/\tau$ and $S = \bar\psi\psi = C_0/\tau$, with C_1 and C_0 being the constant of integration. Putting it in the scalar field equation one obtains

$$
\dot\varphi = (C - \lambda C_1)/\tau, \qquad C = \text{const}. \tag{50}
$$

With all these taken into account, the spinor field equation can be written as

$$
\gamma^0\left(\frac{\partial}{\partial t}+\frac{\dot\tau}{2\tau}\right)\psi + im\psi - \frac{i\lambda(C-\lambda C_1)}{\tau}\gamma^0\psi = 0, \tag{51}
$$

with the solution

$$
\begin{aligned}
\psi_{1,2}(t) &= \frac{D_{1,2}}{\sqrt{\tau}}\exp\left[-i\left\{mt - \lambda(C-\lambda C_1)\int\tau^{-1}dt\right\}\right],\\
\psi_{3,4}(t) &= \frac{D_{3,4}}{\sqrt{\tau}}\exp\left[i\left\{mt + \lambda(C-\lambda C_1)\int\tau^{-1}dt\right\}\right].
\end{aligned}
\tag{52}
$$

The components of energy-momentum tensor in this case read

$$
\begin{aligned}
T_0^0 &= \frac{mC_0}{\tau} + \frac{C}{\tau^2} + \frac{\varepsilon_0}{\tau^{1+\xi}},\\
T_1^1 = T_2^2 = T_3^3 &= -\frac{(C-\lambda C_1)^2}{2\tau^2} - \frac{\varepsilon_0\xi}{\tau^{1+\xi}},
\end{aligned}
$$

where $C := (C^2 - \lambda^2 C_1^2)/2$ and $0 < \xi < 1$. Taking this into account, from (18) and (24) one gets

$$
\tau|_{t\to 0} \approx \sqrt{X/3 - 3\kappa C}\, t \to 0, \qquad \tau|_{t\to\infty} \approx \sqrt{3\kappa m C_0}\, t^2,
$$

which means the solution obtained is initially singular and the isotropization process of the initially anisotropic universe takes place as $t\to\infty$.

Case 3. In this case the scalar and spinor field equations read

$$
\frac{\partial}{\partial t}[\tau(\dot\varphi + i\lambda\bar\psi\gamma^0\gamma^5\psi)] = 0, \tag{53}
$$

$$
\begin{aligned}
&i\gamma^0\left(\frac{\partial}{\partial t}+\frac{\dot\tau}{2\tau}\right)\psi - m\psi + i\lambda\dot\varphi\gamma^0\gamma^5\psi = 0,\\
&i\left(\frac{\partial}{\partial t}+\frac{\dot\tau}{2\tau}\right)\bar\psi\gamma^0 + m\bar\psi - \lambda\dot\varphi\bar\psi\gamma^0\gamma^5 = 0.
\end{aligned}
\tag{54}
$$

We consider the massless spinor field ($m = 0$). In this case from the spinor field equations one finds $i\bar\psi\gamma^0\gamma^5\psi = C_2/\tau$, with C_2 being the constant of integration. Putting it in the scalar field equation one obtains

$$
\dot\varphi = (C - \lambda C_2)/\tau, \qquad C = \text{const.} \tag{55}
$$

With all these taken into account, the spinor field equation can be written as

$$
\gamma^0\left(\frac{\partial}{\partial t}+\frac{\dot\tau}{2\tau}\right)\psi - \frac{i\lambda(C-\lambda C_2)}{\tau}\gamma^0\gamma^5\psi = 0. \tag{56}
$$

Defining $W(t) = \sqrt{\tau}\,\psi(t)$ one writes the foregoing equations as

$$
\begin{aligned}
&\dot W_1 - \lambda\dot\varphi W_3 = 0, \qquad \dot W_2 - \lambda\dot\varphi W_4 = 0,\\
&\dot W_3 - \lambda\dot\varphi W_1 = 0, \qquad \dot W_1 - \lambda\dot\varphi W_3 = 0.
\end{aligned}
\tag{57}
$$

Differentiating the fisrt equation of the foregoing system one gets

$$
\ddot W_1 + \frac{\dot\tau}{\tau}\dot W_1 - [\lambda(C-\lambda C_2)]^2\frac{1}{\tau^2}W_1 = 0, \tag{58}
$$

where the third equation of the system as well as $\dot\varphi$ has been taken into account. The first integral of this equation reads

$$
\tau\dot W_1 = \lambda(C-\lambda C_2)W_1, \tag{59}
$$

with the constant of integration taken to be trivial. Procceding analogically one writes the solution of the system as

$$
\begin{aligned}
W_{1,3} &= D_+ \exp[\lambda(C-\lambda C_2)\int \tau^{-1}dt],\\
W_{2,4} &= D_- \exp[\lambda(C-\lambda C_2)\int \tau^{-1}dt].
\end{aligned}
\tag{60}
$$

The components of energy-momentum tensor in this case read

$$
\begin{aligned}
T_0^0 &= \frac{mC_0}{\tau} + \frac{C}{\tau^2} + \frac{\varepsilon_0}{\tau^{1+\xi}},\\
T_1^1 = T_2^2 = T_3^3 &= -\frac{(C-\lambda C_2)^2}{2\tau^2} - \frac{\varepsilon_0\xi}{\tau^{1+\xi}},
\end{aligned}
$$

where $C := (C^2 - \lambda^2 C_2^2)/2$ and $0 < \xi < 1$. From (18) and (24) in this case one finds

$$
\tau|_{t\to 0} \approx \sqrt{X/3 - 3\kappa C t} \to 0, \qquad \tau|_{t\to\infty} \approx \sqrt{3\kappa m C_0}\, t^2,
$$

which means the solution obtained is initially singular and the isotropization process of the initially anisotropic universe takes place as $t\to\infty$.

4. CONCLUSIONS

Exact solutions to the self-consistent system of spinor and scalar field equations have been obtained for the BI space-time filled with perfect fluid. It is shown that the solutions obtained are initially singular and the space-time is basically asymptotically isotropic independent of the choice of interacting term in the Lagrangian, though there are some special cases that occur initially regular (with breaking energy-dominent condition; Ref. 15) solutions and leave the space-time asymptotically anisotropic.

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