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Nonlinear Spinor Field: Plane-symmetric Solutions

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We consider a nonlinear spinor field in general relativity. The nonlinearity in the spinor field Lagrangian is given by an arbitrary function of the invariants generated from the bilinear spinor forms $S = \bar{\psi}\psi$ and $P = i\bar{\psi}\gamma^5\psi$. Exact plane-symmetric solutions to the gravitational and spinor field equations have been obtained. Role of gravitational field in the formation of the field configurations with limited total energy, spin and charge has been investigated. Influence of the change of the sign of energy density of the spinor field on the properties of the configurations obtained has been examined.

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I. INTRODUCTION

Nonlinear selfcouplings of the spinor fields may arise as a consequence of the geometrical structure of the space-time and, more precisely, because of the existence of torsion. As soon as 1938, Ivanenko [1–3] showed that a relativistic theory imposes in some cases a fourth order selfcoupling. In 1950 Weyl [4] proved that, if the affine and the metric properties of the space-time are taken as independent, the spinor field obeys either a linear equation in a space with torsion or a nonlinear one in a Riemannian space. As the selfaction is of spin-spin type, it allows the assignment of a dynamical role to the spin and offers a clue about the origin of the nonlinearities. This question was further clarified in some important papers by Utiyama, Kibble and Sciama [5–7] In the simplest scheme the selfaction is of pseudovector type, but it can be shown that one can also get a scalar coupling [8]. An excellent review of the problem may be found in [9,10].

Nonlinear quantum Dirac fields were used by Heisenberg [11,12] in his ambitious unified theory of elementary particles. They are presently the object of renewed interest since the widely known paper by Gross and Neveu [13]

Nonlinear spinor field (NLSF) in external cosmological gravitational field was first studied by Shikin in 1991 [14]. This study was extended by us for the more general case where we consider the nonlinear term as an arbitrary function of all possible invariants generated from spinor bilinear forms. In that paper we also studied the possibility of elimination of initial singularity especially for the Kasner universe [15]. For few years we studied the behavior of self-consistent NLSF in a BI universe [16,17] both in presence of perfect fluid and without it that was followed by the Refs. [18–20], where we studied the self-consistent system of interacting spinor and scalar fields. A detail review of nonlinear spinor field in BI universe can be found in [21]. In a series of paper we also thoroughly studied the interacting scalar and electromagnetic fields in spherically and cylindrically space-time [22–26]. The purpose of the paper is to study the role of slef gravitation in the formation of configurations with localized energy density and limited total energy, spin and charge of the spinor field.

II. FUNDAMENTAL EQUATIONS AND GENERAL SOLUTIONS

The Lagrangian of the nonlinear spinor and gravitational fields can be written in the form

$$L = \frac{R}{2\kappa} + L_{sp}, \tag{2.1}$$

with

$$L_{sp} = \frac{i}{2}\Big[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi\Big] - m\bar\psi\psi + L_N. \tag{2.2}$$

Here R is the scalar curvature and κ is the Einstein's gravitational constant. The nonlinear term L_N in spinor Lagrangian describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field having the form

$$S = \bar\psi\psi, \quad P = i\bar\psi\gamma^5\psi, \quad v^\mu = (\bar\psi\gamma^\mu\psi), \quad A^\mu = (\bar\psi\gamma^5\gamma^\mu\psi), \quad T^{\mu\nu} = (\bar\psi\sigma^{\mu\nu}\psi),$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$. Invariants, corresponding to the bilinear forms, look like

$$I = S^2, \quad J = P^2, \quad I_v = v_\mu\, v^\mu = (\bar\psi\gamma^\mu\psi)\, g_{\mu\nu}(\bar\psi\gamma^\nu\psi),$$

$$I_A = A_\mu\, A^\mu \;=\; (\bar\psi \gamma^5 \gamma^\mu \psi)\, g_{\mu\nu} (\bar\psi \gamma^5 \gamma^\nu \psi), \quad I_T = T_{\mu\nu}\, T^{\mu\nu} \;=\; (\bar\psi \sigma^{\mu\nu} \psi)\, g_{\mu\alpha} g_{\nu\beta} (\bar\psi \sigma^{\alpha\beta} \psi).$$

According to the Pauli-Fierz theorem, [28] among the five invariants only I and J are independent as all other can be expressed by them: $I_v = -I_A = I + J$ and $I_T = I - J$. Therefore we choose the nonlinear term $L_N = F(I, J)$, thus claiming that it describes the nonlinearity in the most general of its form.

The static plane-symmetric metric we choose in the form

$$ds^2 = e^{2\rho} dt^2 - e^{2\alpha} dx^2 - e^{2\beta} (dy^2 + dz^2), \tag{2.3}$$

where the metric functions ρ, α, β depend on the spatial variable x only and obey the coordinate condition

$$\alpha = 2\beta + \rho. \tag{2.4}$$

Variation of (2.1) with respect to spinor field $\psi\,(\bar\psi)$ gives nonlinear spinor field equations

$$i\gamma^\mu \nabla_\mu \psi - \Phi \psi + i\mathcal{G} \gamma^5 \psi \;=\; 0, \tag{2.5a}$$

$$i\nabla_\mu \bar\psi \gamma^\mu + \Phi \bar\psi - i\mathcal{G} \bar\psi \gamma^5 \;=\; 0, \tag{2.5b}$$

with

$$\Phi = m - \mathcal{D} = m - 2S\frac{\partial F}{\partial I}, \qquad \mathcal{G} = 2P\frac{\partial F}{\partial J}.$$

Varying (2.1) with respect to metric tensor $g_{\mu\nu}$ we obtain the Einstein's field equation

$$R_\nu^\mu - \frac{1}{2}\,\delta_\nu^\mu R = -\kappa T_\nu^\mu \tag{2.6}$$

which in view of (2.3) and (2.4) is written as follows

$$G_0^0 = e^{-2\alpha}(2\beta - 2\rho\beta - \beta^2) = -\kappa T_0^0 \tag{2.7a}$$

$$G_1^1 = e^{-2\alpha}(2\rho\beta + \beta^2) = -\kappa T_1^1 \tag{2.7b}$$

$$G_2^2 = e^{-2\alpha}(\beta + \rho - 2\rho\beta - \beta^2) = -\kappa T_2^2 \tag{2.7c}$$

$$G_3^3 = G_2^2, \quad T_3^3 = T_2^2. \tag{2.7d}$$

Here prime denotes differentiation with respect to x and T_ν^μ is the energy-momentum tensor of the spinor field

$$T_\mu^\rho = \frac{i}{4} g^{\rho\nu} \Bigl(\bar\psi \gamma_\mu \nabla_\nu \psi + \bar\psi \gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar\psi \gamma_\nu \psi - \nabla_\nu \bar\psi \gamma_\mu \psi\Bigr) - \delta_\mu^\rho L_{sp} \tag{2.8}$$

where L_{sp} with respect to (2.5) takes the form

$$L_{sp} = -\frac{1}{2}\Bigl(\bar\psi\frac{\partial F}{\partial \bar\psi} + \frac{\partial F}{\partial \psi}\psi\Bigr) - F. \tag{2.9}$$

In (2.5) and (2.8) ∇_μ denotes the covariant derivative of spinor, having the form [29,30]

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \tag{2.10}$$

where $\Gamma_\mu(x)$ are spinor affine connection matrices. γ matrices in the above equations are connected with the flat space-time Dirac matrices $\bar{\gamma}$ in the following way

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}, \quad \gamma_\mu(x) = e^a_\mu(x) \bar{\gamma}_a, \tag{2.11}$$

where $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ and e^a_μ is a set of tetrad 4-vectors. Using (2.11) we obtain

$$\gamma^0(x) = e^{-\rho}\bar{\gamma}^0, \quad \gamma^1(x) = e^{-\alpha}\bar{\gamma}^1, \quad \gamma^2(x) = e^{-\beta}\bar{\gamma}^2, \quad \gamma^3(x) = e^{-\beta}\bar{\gamma}^3. \tag{2.12}$$

From

$$\Gamma_\mu(x) = \frac{1}{4} g_{\rho\sigma}(x)\Big(\partial_\mu e^b_\delta e^\rho_b - \Gamma^\rho_{\mu\delta}\Big)\gamma^\sigma\gamma^\delta, \tag{2.13}$$

one finds

$$\Gamma_0 = -\frac{1}{2}\bar{\gamma}^0\bar{\gamma}^1 e^{-2\beta}\rho, \quad \Gamma_1 = 0, \quad \Gamma_2 = \frac{1}{2}\bar{\gamma}^2\bar{\gamma}^1 e^{-(\rho+\beta)}\beta, \quad \Gamma_3 = \frac{1}{2}\bar{\gamma}^3\bar{\gamma}^1 e^{-(\rho+\beta)}\beta. \tag{2.14}$$

Flat space-time matrices $\bar{\gamma}$ we will choose in the form, given in [31]:

$$\bar{\gamma}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \bar{\gamma}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\gamma}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\gamma}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Defining γ^5 as follows,

$$\gamma^5 = -\frac{i}{4} E_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho, \quad E_{\mu\nu\sigma\rho} = \sqrt{-g}\,\varepsilon_{\mu\nu\sigma\rho}, \quad \varepsilon_{0123} = 1,$$
$$\gamma^5 = -i\sqrt{-g}\gamma^0\gamma^1\gamma^2\gamma^3 = -i\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3 = \bar{\gamma}^5,$$

we obtain

$$\bar{\gamma}^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We will consider the spinor field to be the function of the spatial coordinate x only $[\psi = \psi(x)]$. Using (2.10), (2.12) and (2.14) we find

$$\gamma^\mu \Gamma_\mu = -\frac{1}{2} e^{-\alpha} \alpha \bar{\gamma}^1. \tag{2.15}$$

Then taking into account (2.15) we rewrite the spinor field equation (2.5a) as

$$i\bar{\gamma}^1 \Bigl(\frac{\partial}{\partial x} + \frac{\alpha}{2}\Bigr)\psi + ie^{\alpha}\Phi\psi + e^{\alpha}\mathcal{G}\gamma^5\psi = 0. \tag{2.16}$$

Further setting $V(x) = e^{\alpha/2}\psi(x)$ with

$$V(x) = \begin{pmatrix} V_1(x) \\ V_2(x) \\ V_3(x) \\ V_4(x) \end{pmatrix}$$

for the components of spinor field from (2.16) one deduces the following system of equations:

$$V_4 + ie^{\alpha}\Phi V_1 - e^{\alpha}\mathcal{G} V_3 = 0, \tag{2.17a}$$
$$V_3 + ie^{\alpha}\Phi V_2 - e^{\alpha}\mathcal{G} V_4 = 0, \tag{2.17b}$$
$$V_2 - ie^{\alpha}\Phi V_3 + e^{\alpha}\mathcal{G} V_1 = 0, \tag{2.17c}$$
$$V_1 - ie^{\alpha}\Phi V_4 + e^{\alpha}\mathcal{G} V_2 = 0. \tag{2.17d}$$

As one sees, the equation (2.17) gives following relations

$$V_1^2 - V_2^2 - V_3^2 + V_4^2 = \text{const}. \tag{2.18}$$

Using the solutions obtained one can write the components of spinor current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{2.19}$$

Taking into account that $\bar{\psi} = \psi^\dagger \bar{\gamma}^0$, where $\psi^\dagger = (\psi_1^*,\, \psi_2^*,\, \psi_3^*,\, \psi_4^*)$ and $\psi_j = e^{-\alpha/2} V_j, \quad j = 1,2,3,4$ for the components of spinor current we write

$$j^0 = [V_1^* V_1 + V_2^* V_2 + V_3^* V_3 + V_4^* V_4] e^{-(\alpha+\rho)}, \tag{2.20a}$$
$$j^1 = [V_1^* V_4 + V_2^* V_3 + V_3^* V_2 + V_4^* V_1] e^{-2\alpha}, \tag{2.20b}$$
$$j^2 = -i[V_1^* V_4 - V_2^* V_3 + V_3^* V_2 - V_4^* V_1] e^{-(\alpha+\beta)}, \tag{2.20c}$$
$$j^3 = [V_1^* V_3 - V_2^* V_4 + V_3^* V_1 - V_4^* V_2] e^{-(\alpha+\beta)}. \tag{2.20d}$$

Since we consider the field configuration to be static one, the spatial components of spinor current vanishes, i.e.,

$$j^1 = 0, \quad j^2 = 0, \quad j^3 = 0. \tag{2.21}$$

This supposition gives additional relation between the constant of integration. The component j^0 defines the charge density of spinor field that has the following chronometric-invariant form

$$\varrho = (j_0 \cdot j^0)^{1/2}. \tag{2.22}$$

The total charge of spinor field is defined as

$$Q = \int\limits_{-\infty}^{\infty} \varrho \sqrt{-^3 g} dx \tag{2.23}$$

(in (2.23) integrations by y and z are performed in the limit $(0, 1)$).

Let us consider the spin tensor [31]

$$S^{\mu\nu,\epsilon} = \frac{1}{4}\bar{\psi}\{\gamma^\epsilon \sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\epsilon\}\psi. \tag{2.24}$$

We write the components $S^{ik,0}$ $(i, k = 1, 2, 3)$, defining the spatial density of spin vector explicitly. From (2.24) we have

$$S^{ij,0} = \frac{1}{4}\bar{\psi}\{\gamma^0 \sigma^{ij} + \sigma^{ij}\gamma^0\}\psi = \frac{1}{2}\bar{\psi}\gamma^0 \sigma^{ij}\psi \tag{2.25}$$

that defines the projection of spin vector on k axis. Here i, j, k takes the value 1, 2, 3 and $i \neq j \neq k$. Thus, for the projection of spin vectors on the X, Y and Z axis we find

$$S^{23,0} = [V_1^* V_2 + V_2^* V_1 + V_3^* V_4 + V_4^* V_3] e^{-\alpha - 2\beta - \rho}, \tag{2.26a}$$
$$S^{31,0} = [V_1^* V_2 - V_2^* V_1 + V_3^* V_4 - V_4^* V_3] e^{-2\alpha - \beta - \rho}, \tag{2.26b}$$
$$S^{12,0} = [V_1^* V_1 - V_2^* V_2 + V_3^* V_3 - V_4^* V_4] e^{-2\alpha - \beta - \rho}. \tag{2.26c}$$

The chronometric invariant spin tensor takes the form

$$S_{\rm ch}^{ij,0} = (S_{ij,0} S^{ij,0})^{1/2}, \tag{2.27}$$

and the projection of the spin vector on k axis is defined by

$$S_k = \int\limits_{-\infty}^{\infty} S_{\rm ch}^{ij,0} \sqrt{-^3 g} dx. \tag{2.28}$$

(In (2.28), as well as in (2.23) integrations by y and z are performed in the limit $(0, 1)$).

From (2.5) one can write the equations for $S = \bar{\psi}\psi$, $\quad P = i\bar{\psi}\gamma^5\psi$ and $A = \bar{\psi}\bar{\gamma}^5\bar{\gamma}^1\psi$

$$S + \alpha S + 2e^\alpha \mathcal{G}\, A = 0, \tag{2.29a}$$
$$P + \alpha P + 2e^\alpha \Phi\, A = 0, \tag{2.29b}$$
$$A + \alpha A + 2e^\alpha \Phi\, P + 2e^\alpha \mathcal{G} S = 0. \tag{2.29c}$$

Note that, A in (2.29) is indeed the pseudo-vector A^1. Here for simplicity, we use the notation A. From (2.29) immediately follows

$$S^2 + P^2 - A^2 = C_0 e^{-2\alpha}, \quad C_0 = \text{const}. \tag{2.30}$$

Let us now solve the Einstein equations. To do it we first write the expression for the components of the energy-momentum tensor explicitly. Using the property of flat space-time Dirac matrices and the explicit form of covariant derivative ∇_μ, for the spinor field one finds

$$T_1^1 = m\, S \,-\, F(I,\, J), \quad T_0^0 = T_2^2 = T_3^3 = \mathcal{D}S + \mathcal{G}P - F(I,\, J). \tag{2.31}$$

In view of $T_0^0 = T_2^2$, subtraction of Einstein equations (2.7a) and (2.7c) leads to the equation

$$\beta - \gamma = 0, \tag{2.32}$$

with the solution

$$\beta(x) = \gamma(x) + Bx, \tag{2.33}$$

where B is the integration constant. The second constant has been chosen to be trivial, since it acts on the scale of Y and Z axes only. In account of (2.32) from (2.4) one obtains

$$\beta = \frac{1}{3}\alpha, \quad \gamma = \frac{1}{3}\alpha. \tag{2.34}$$

Solutions to the equation (2.34) together with (2.4) and (2.33) lead to the following expression for $\beta(x)$ and $\gamma(x)$

$$\beta(x) = \frac{1}{3}(\alpha(x) + BX), \quad \gamma(x) = \frac{1}{3}(\alpha(x) - 2Bx). \tag{2.35}$$

Equation (2.7b), being the first integral of (2.7a) and (2.7c), is a first order differential equation. Inserting β and γ from (2.35) and T_1^1 in account of (2.31) into (2.7b) for α one gets

$$\alpha^2 - B^2 = -3\kappa e^{2\alpha}\Big[mS - F(I, J)\Big]. \tag{2.36}$$

AS one sees from (2.29) and (2.30), the invariants are the functions of α, so is the right hand side of (2.36), hence can be solved in quadrature. In the sections to follow, we analyze the equation (2.36) in details given the concrete form of nonlinear term in spinor Lagrangian.

III. ANALYSIS OF THE RESULTS

In this section we shall analyze the general results obtained in the previous section for concrete nonlinear term.

A. Case with linear spinor field

Let us first consider the linear spinor field. By doing so we can compare the results obtained with those for nonlinear spinor one, hence clarify the role of nonlinearity of the fields in question in the formation of regular localized solutions such as static solitary wave or solitons [33,34].

In this case for the linear spinor field we have

$$T_1^{\ 1} = m\,S, \quad T_0^{\ 0} = T_2^{\ 2} = T_3^{\ 3} = 0. \tag{3.1}$$

As one can easily verify, for the linear spinor field the equation (2.29a) results

$$S = C_0 e^{-\alpha}. \tag{3.2}$$

Taking this relation into account and the fact that $\alpha(x) = -\frac{1}{S}\frac{dS}{dx}$ from (2.36) we write

$$\int \frac{dS}{\sqrt{B^2 S^2 - 3\kappa m C_0^2 S}} = x, \tag{3.3}$$

with the solution

$$S(x) = \frac{M^2}{B^2}\cosh^2(\tilde{H}x), \quad M^2 = 3\kappa m C_0^2, \quad \tilde{H} = B/2. \tag{3.4}$$

Further we define the functions ψ_j. Taking into account that in this case

$$\mathcal{F}(S) = mC_0/S\sqrt{B^2S^2 - M^2S},$$

for $N_{1,2}$ in view of (3.4) we find

$$N_{1,2}(x) = \pm(2B/3\kappa C_0)\tanh(\tilde{H}x) + R_{1,2}.$$

We can then finally write

$$\begin{aligned} \psi_{1,2}(x) &= i a_{1,2} E(x)\cosh[f(x) + R_{1,2}], \\ \psi_{3,4}(x) &= a_{2,1} E(x)\sinh[f(x) + R_{2,1}], \end{aligned} \tag{3.5}$$

where $E(x) = \sqrt{3\kappa m C_0/B^2}\cosh(\tilde{H}x)$ and $f(x) = (2B/3\kappa m C_0)\tanh(\tilde{H}x)$.

For the total charge Q of the system in this case we have

$$Q = 2a^2 \int\limits_{-\infty}^{\infty} \cosh\Big[\frac{4B}{3\kappa C_0}\tanh(\tilde{H}x) + 2R\Big]\Big(\frac{C_0 B^2}{M^2\cosh^2(\tilde{H}x)}\Big)^{3/2} e^{2Bx/3}\,dx < \infty. \tag{3.6}$$

It can be shown that, in case of linear spinor field both charge and spin of spinor field are limited.

B. Nonlinear spinor field

Case I: F = F(I). Let us consider the case when the nonlinear term in spinor field Lagrangian is a function of I (S) only, that leads to $\mathcal{G} = 0$. From (2.29) as in case of linear spinor field we find $S = C_0 e^{-\alpha(x)}$. Proceeding as in foregoing subsection, for S from (2.36) we write

$$\frac{dS}{dx} = \pm\sqrt{B^2 S^2 - 3\kappa C_0^2\Big[mS - F(S)\Big]} \tag{3.7}$$

with the solution

$$\int \frac{dS}{\sqrt{B^2 S^2 - 3\kappa C_0^2\Big[mS - F(S)\Big]}} = \pm(x + x_0). \tag{3.8}$$

Given the concrete form of the functions $F(S)$ from (3.8) yields S, hence α, β, ρ.

Let us now go back to spinor field equations (2.17). Setting $V_j(x) = U_j(S)$, $j = 1,2,3,4$ and taking into account that in this case $\mathcal{G} = 0$, for $U_j(S)$ we obtain

$$\frac{dU_4}{dS} + i\mathcal{F}(S)U_1 = 0, \tag{3.9a}$$
$$\frac{dU_3}{dS} + i\mathcal{F}(S)U_2 = 0, \tag{3.9b}$$
$$\frac{dU_2}{dS} - i\mathcal{F}(S)U_3 = 0, \tag{3.9c}$$
$$\frac{dU_1}{dS} - i\mathcal{F}(S)U_4 = 0, \tag{3.9d}$$

with $\mathcal{F}(S) = \Phi\mathcal{L}(S)C_0/S$. Differentiating (3.9a) with respect to S and inserting (3.9d) into it for U_4 we find

$$\frac{d^2U_4}{dS^2} - \frac{1}{\mathcal{F}}\frac{d\mathcal{F}}{dS}\frac{dU_4}{dS} - \mathcal{F}^2 U_4 = 0 \tag{3.10}$$

that transforms to

$$\frac{1}{\mathcal{F}}\frac{d}{dS}\Big(\frac{1}{\mathcal{F}}\frac{dU_4}{dS}\Big) - U_4 = 0, \tag{3.11}$$

with the first integral

$$\frac{dU_4}{dS} = \pm\sqrt{U_4^2 + C_1}\cdot\mathcal{F}(S), \quad C_1 = \text{const}. \tag{3.12}$$

For $C_1 = a_1^2 > 0$ from (3.12) we obtain

$$U_4(S) = a_1 \sinh N_1(S), \quad N_1 = \pm\int\mathcal{F}(S)dS + R_1, \quad R_1 = \text{const}. \tag{3.13}$$

whereas, for $C_1 = -b_1^2 < 0$ from (3.12) we obtain

$$U_4(S) = a_1 \cosh N_1(S) \tag{3.14}$$

Inserting (3.13) and (3.14) into (3.9d) one finds

$$U_1(S) = ia_1 \cosh N_1(S), \quad U_1(S) = ib_1 \sinh N_1(S). \tag{3.15}$$

Analogically, for U_2 and U_3 we obtain

$$U_3(S) = a_2 \mathrm{sinh} N_2(S), \quad U_3(S) = b_2 \mathrm{cosh} N_2(S). \tag{3.16}$$

and

$$U_2(S) = ia_2 \mathrm{cosh} N_2(S), \quad U_2(S) = ib_2 \mathrm{sinh} N_2(S). \tag{3.17}$$

where $N_2 = \pm \int \mathcal{F}(S) dS + R_2$ and a_2, b_2 and R_2 are the integration constants. Thus we find the general solutions to the spinor field equations (3.9) containing four arbitrary constants.

Using the solutions obtained, from (2.20) we find the components of spinor current

$$j^0 = [a_1^2 \mathrm{cosh}(2N_1(S)) + a_2^2 \mathrm{cosh}(2N_2(S))] e^{-(\alpha+\rho)}, \tag{3.18a}$$
$$j^1 = 0, \tag{3.18b}$$
$$j^2 = -[a_1^2 \mathrm{sinh}(2N_1(S)) - a_2^2 \mathrm{sinh}(2N_2(S))] e^{-(\alpha+\beta)}, \tag{3.18c}$$
$$j^3 = 0. \tag{3.18d}$$

The supposition (2.21) leads to the following relations between the constants: $a_1 = a_2 = a$ and $R_1 = R_2 = R$, since $N_1(S) = N_2(S) = N(S)$. The chronometric-invariant form of the charge density and the total charge of spinor field are

$$\varrho = 2a^2 \mathrm{cosh}(2N(S)) e^{-\alpha}, \tag{3.19}$$

$$Q = 2a^2 \int\limits_{-\infty}^{\infty} \mathrm{cosh}(2N(S)) e^{\alpha-\rho} dx. \tag{3.20}$$

From (2.25) we find

$$S^{12,0} = 0, \quad S^{13,0} = 0, \quad S^{23,0} = a^2 \mathrm{cosh}(2N(S)) e^{-2\alpha}. \tag{3.21}$$

Thus, the only nontrivial component of the spin tensor is $S^{23,0}$ that defines the projection of spin vector on X axis. From (2.27) we write the chronometric invariant spin tensor

$$S_{\mathrm{ch}}^{23,0} = a^2 \mathrm{cosh}(2N(S)) e^{-\alpha}, \tag{3.22}$$

and the projection of the spin vector on X axis

$$S_1 = a^2 \int\limits_{-\infty}^{\infty} \mathrm{cosh}(2N(S)) e^{\alpha-\rho} dx. \tag{3.23}$$

(in (2.28), as well as in (2.23) integrations by y and z are performed in the limit $(0,1)$). Note that the integrants both in (3.20) and (3.23) coincide.

Let us now analyze the result obtained choosing the nonlinear term in the form $F(I) = \lambda S^n = \lambda I^{n/2}$ with $n \geq 2$ and λ is the parameter of nonlinearity. For $n = 2$ we have Heisenberg-Ivanenko type nonlinear spinor field equation [35]

$$ie^{-\alpha} \bar{\gamma}^1 (\partial_x + \frac{1}{2}\alpha)\psi - m\psi + 2\lambda(\bar{\psi}\psi)\psi = 0. \tag{3.24}$$

Setting $F = S^2$ into (3.8) we come to the expression for S that is similar to that for linear case with

$$B^2 \to H_1^2 = B^2 + 3\kappa\lambda C_0. \tag{3.25}$$

Let us write the functions ψ_j explicitly. In this case we have

$$\mathcal{F}(S) = m(C_0 - 2\lambda S)/S\sqrt{H_1^2 S^2 - M^2 S},$$

and

$$N_{1,2}(x) = (2H_1/3\kappa C_0)\mathrm{tanh}(\bar{H}_1 x) - 2\lambda C_0 x + R_{1,2}, \quad \bar{H}_1 = H_1/2.$$

We can then finally write

$$\begin{aligned} \psi_{1,2}(x) &= ia_{1,2}\frac{\sqrt{3\kappa m C_0}}{H_1}\mathrm{cosh}(\bar{H}_1 x)\mathrm{cosh}N_{1,2}(x), \\ \psi_{3,4}(x) &= ia_{2,1}\frac{\sqrt{3\kappa m C_0}}{H_1}\mathrm{cosh}(\bar{H}_1 x)\mathrm{cosh}N_{2,1}(x). \end{aligned} \tag{3.26}$$

The energy-density distribution of the spinor field in the case considered in a unit invariant volume is

$$T_0^0(-{}^3g)^{1/2} = \lambda C_0^2\Big[\frac{3\kappa m C_0}{H_1^2}\mathrm{cosh}^2(\bar{H}_1 x)\Big]^{1/3} e^{2Bx/3}. \tag{3.27}$$

From (3.27) follows that, the energy density of the system is not localized and the Heisenberg-Ivanenko equation does not possess soliton-like solutions [36].

In case of $n > 2$, the energy density of the system in question is

$$T_0^0 = \lambda(n-1)S^n. \tag{3.28}$$

The equation (3.28) shows that the regular solutions with localized energy density exists iff $S = \bar{\psi}\psi$ is a continuous and limited function and $\lim\limits_{x\to\pm\infty} S(x) \to 0$. Inserting $F(I) = \lambda S^n, \quad n > 2$ into (3.8) we find the condition, when S possesses the properties mentioned above:

$$\int\frac{dS}{\sqrt{B^2 S^2 - 3\kappa C_0^2(mS - \lambda S^n)}} = x. \tag{3.29}$$

As one sees from (3.29), for $m \neq 0$ at no value of x S becomes trivial, since as $S \to 0$, the denominator of the integrant beginning from some finite value of S becomes imaginary. It means that for $S(x)$ to be trivial at spatial infinity ($x \to \infty$), it is necessary to choose massless spinor field setting $m = 0$ in (3.29). Note that, in the unified nonlinear spinor theory of Heisenberg, the massive term is absent, and according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [37]. It should be emphasized that in the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system [38]. Thus without losing the generality we can consider massless spinor field putting $m = 0$. Note that in the sections to follow where we consider the nonlinear spinor term as $F = P^n$, or $F = (K_\pm)^n$ with $K_\pm = (I \pm J)$, we will study the massless spinor field only.

From (3.29) for $m = 0$, $\lambda > 0$ and $n > 2$ for $S(x)$ we obtain

$$S(x) = \Big[-H_1/\sqrt{3\kappa\lambda C_0^2(\zeta^2 - 1)}\Big]^{2/(n-2)}, \quad \zeta = \cosh[(n-2)\bar{H}_1 x] \tag{3.30}$$

from which follows that $\lim\limits_{x\to 0}|S(x)| \to \infty$. It means that $T_0^0(x)$ is not bounded at $x = 0$ and the initial system of equations does not possess solutions with localized energy density.

If we set in (3.29) $m = 0$, $\lambda = -\Lambda^2 < 0$ and $n > 2$, then for S we obtain

$$S(x) = \Big[H_1/\sqrt{3\kappa\lambda C_0^2 \zeta}\Big]^{2/(n-2)} \tag{3.31}$$

It is seen from (3.31) that $S(x)$ has maximum at $x = 0$ and $\lim\limits_{x\to\pm\infty} S(x) \to 0$. Taking into account the energy density

$$T_0^0 = -\Lambda^2(n-1)S^n, \tag{3.32}$$

of the case in question, one can show that for $B > 0$ and $n > 7/3$, the spinor field energy-density is localized in the space along x-axis and the total field energy (if integration limits along y and z axes are finit) is bound, i.e.,

$$|E| = |\int\limits_{-\infty}^{\infty} T_0^0 \sqrt{-{}^3 g}\, dx| < \infty.$$

The components of spinor field in this case have the form

$$\begin{aligned} \psi_{1,2}(x) &= i a_{1,2} E(x) \cosh N_{1,2}(x), \\ \psi_{3,4}(x) &= a_{2,1} E(x) \sinh N_{2,1}(x), \end{aligned} \tag{3.33}$$

where

$$E(x) = (1/\sqrt{C_0})\Big[H_1/\sqrt{3\kappa\Lambda^2 C_0^2 \zeta}\Big]^{1/(n-2)}$$

and

$$N_{1,2}(x) = -\frac{2nH_1\sqrt{\zeta^2 - 1}}{3\kappa C_0(n-2)\zeta} + R_{1,2}.$$

For the solutions obtained we write the chronometric-invariant charge density of the spinor field ϱ:

$$\varrho(x) = \frac{2a^2}{C_0}\cosh\Big\{-\frac{4nH_1\sqrt{\zeta^2 - 1}}{3\kappa C_0(n-2)\zeta} + 2R\Big\}\Big\{\frac{H_1^2}{3\kappa\Lambda^2 C_0^2 \zeta^2}\Big\}^{1/(n-2)}. \tag{3.34}$$

As one sees from (3.34), the charge density is localized, since $\lim\limits_{x\to\pm\infty} \varrho(x) \to 0$. Nevertheless, the charge density of the spinor field, coming to unit invariant volume $\varrho\sqrt{-{}^3 g}$, is not localized:

$$\varrho\sqrt{-{}^3 g} = 2a^2 \cosh[2N(x)] e^{\alpha - \gamma} = 2a^2 \cosh[2N(x)](C_0/S)^{2/3} e^{2Bx/3}. \tag{3.35}$$

It leads to the fact that the total charge of the spinor field is not bounded as well. As far as the expression for chronometric-invariant tensor of spin (3.22) coincides with that of $\varrho(x)/2$, the conclusions made for $\varrho(x)$ and Q will be valid for the spin tensor $S_{\rm ch}^{23,0}$ and projection of spin vector on X axis S_1, i.e., $S_{\rm ch}^{23,0}$ is localized and S_1 is unlimited.

The solution obtained describes the configuration of nonlinear spinor field with localized energy density but with the metric that is singular at spatial infinity, as in this case

$$e^{2\alpha} = (C_0/S)^2 = C_0^2\Bigl\{\frac{3\kappa\Lambda C_0^2\zeta}{H_1^2}\Bigr\}^{2/(n-2)}\Bigr|_{x\to\pm\infty} \to \infty \tag{3.36}$$

Let us consider the massless spinor field with

$$F = -\Lambda^2 S^{-\nu}, \quad \nu = \text{constant} > 0. \tag{3.37}$$

In this case the energy density of the nonlinear spinor field takes the form

$$T_0^0 = \Lambda^2(\nu+1)S^{-\nu} \tag{3.38}$$

For S in this case we get

$$\int\frac{dS}{\sqrt{B^2S^2 - 3\kappa C_0^2\Lambda^2 S^{-\nu}}} = x \tag{3.39}$$

with the solution

$$S(x) = \Bigl[\frac{3\kappa\Lambda^2 C_0^2}{H_1^2}\zeta_1^2\Bigr]^{1/(\nu+2)}, \quad \zeta_1 = \cosh[(\nu+2)\bar{H}_1 x]. \tag{3.40}$$

For energy density in this case we have

$$T_0^0(x) = \Lambda^2(\nu+1)\Bigl[\frac{H_1^2}{3\kappa C_0^2\Lambda^2\zeta_1^2}\Bigr]^{\nu/(\nu+2)}. \tag{3.41}$$

It follows from (3.41) that the contribution of the spinor field in the energy density is localized.

The energy density distribution of the field system, coming to unit invariant volume is

$$\begin{aligned}\varepsilon(x) &= T_0^0\sqrt{-{}^3g} = \bigl[\Lambda^2(\nu+1)S^{-\nu}\bigr]e^{2\alpha-\gamma}\\ &= \frac{H_1^2(\nu+1)}{3\kappa\zeta_1^2}\Bigl\{\frac{H_1^2}{3\kappa C_0^2\Lambda^2\zeta_1^2}\Bigr\}^{1/3(\nu+2)}e^{2Bx/3}.\end{aligned} \tag{3.42}$$

As one sees from (3.42) $\varepsilon(x)$ is a localized function, i.e., $\lim\limits_{x\to\pm\infty}\varepsilon(x)\to 0$, if $H > 2B$. In this case the total energy is also finite.

The components of spinor field in this case have the form

$$\begin{aligned}\psi_{1,2}(x) &= ia_{1,2}E(x)\cosh N_{1,2}(x),\\ \psi_{3,4}(x) &= a_{2,1}E(x)\sinh N_{2,1}(x),\end{aligned} \tag{3.43}$$

where

$$E(x) = (1/\sqrt{C_0})\Big[\frac{\sqrt{3\kappa\Lambda^2 C_0^2}}{H_1^2}\zeta_1\Big]^{1/(\nu+2)}$$

and

$$N_{1,2}(x) = -\frac{2H\nu\sqrt{\zeta_1^2-1}}{3\kappa C_0(\nu+2)\zeta_1} + R_{1,2}.$$

The chronometric-invariant charge density of the spinor field coming to unit invariant volume with $a_1 = a_2 = a$ and $N_1 = N_2$ reads

$$\begin{aligned} \varrho\sqrt{-{}^3g} &= 2a^2\cosh[2N(x)]e^{\alpha-\gamma} = \\ &= 2a^2(C_0)^{2/3}\cosh\Big\{2R - \frac{4H_1\nu\sqrt{\zeta_1^2-1}}{3\kappa C_0(\nu+2)\zeta_1}\Big\}\Big\{\frac{H_1^2}{3\kappa C_0^2\Lambda^2\zeta_1^2}\Big\}^{2/3(\nu+2)} e^{2Bx/3}. \end{aligned} \tag{3.44}$$

It follows from (3.44) that $\varrho\sqrt{-{}^3g}$ is a localized function and the total charge Q is finite. The spin of spinor field is limited as well.

It should be emphasized that in a flat space-time where $\alpha = \beta = \rho 0$ we have

$$\psi_\mu(x) = C_{\mu 1}e^{Mx} + C_{\mu 2}e^{-Mx}, \quad \mu = 1,2,3,4, \tag{3.45}$$

with $M = m - F' = mC_0$. The integration constants in (3.45) are connected with each other as $C_{31} = -iC_{21}$, $C_{32} = iC_{22}$, $C_{41} = -iC_{11}$, $C_{42} = iC_{12}$. In this case $S = \bar\psi\psi = 4(C_{11}C_{12} + C_{21}C_{22})$ and $F' = C_0$. Hence follows that the nonlinear spinor field not not possess soliton-like solutions in a flat space-time.

Case II: F = F(J). Here we consider the massless spinor field with the nonlinearity $F = F(J)$. In this case from (2.29b) immediately follows

$$P = D_0 e^{-\alpha(x)}, \quad D_0 = \text{const.} \tag{3.46}$$

From (2.17) we now have

$$V_4 - e^{\alpha}\mathcal{G}V_3 = 0, \tag{3.47a}$$
$$V_3 - e^{\alpha}\mathcal{G}V_4 = 0, \tag{3.47b}$$
$$V_2 + e^{\alpha}\mathcal{G}V_1 = 0, \tag{3.47c}$$
$$V_1 + e^{\alpha}\mathcal{G}V_2 = 0, \tag{3.47d}$$

with the solutions

$$V_1 = C_1\sinh[-\mathcal{A} + C_2] \tag{3.48a}$$
$$V_2 = C_1\cosh[-\mathcal{A} + C_2] \tag{3.48b}$$
$$V_3 = C_3\sinh[\mathcal{A} + C_4] \tag{3.48c}$$
$$V_4 = C_3\cosh[\mathcal{A} + C_4] \tag{3.48d}$$

with C_1, C_2, C_3 and C_3 being the constant of integration and $\mathcal{A} = \int e^{\alpha}\mathcal{G}dx$.

Using the solutions obtained, from (2.20) we now find the components of spinor current

$$j^0 = [C_1^2 \cosh[2(-\mathcal{A} + C_2)] + C_3^2 \cosh[2(\mathcal{A} + C_4)]]e^{-(\alpha+\rho)}, \tag{3.49a}$$
$$j^1 = [2C_1C_3 \sinh(C_2 + C_4)]e^{-2\alpha}, \tag{3.49b}$$
$$j^2 = 0, \tag{3.49c}$$
$$j^3 = -[2C_1C_3 \cosh[2\mathcal{A} - C_2 + C_4)]e^{-(\alpha+\beta)}. \tag{3.49d}$$

The supposition (2.21) that the spatial components of the spinor current are trivial leads at least one of the constants (C_1, C_3) to be zero. Let us set $C_1 = 0$. The chronometric-invariant form of the charge density and the total charge of spinor field are

$$\varrho = C_3^2 \cosh[2(\mathcal{A} + C_4)]e^{-\alpha}, \tag{3.50}$$

$$Q = C_3^2 \int\limits_{-\infty}^{\infty} \cosh[2(\mathcal{A} + C_4)]e^{\alpha-\rho} dx. \tag{3.51}$$

From (2.25) we find

$$S^{12,0} = -C_3^2 e^{-(2\alpha+\beta+\rho)}, \quad S^{31,0} = 0, \quad S^{23,0} = C_3^2 \sinh[2(\mathcal{A} + C_4)]e^{-2\alpha}. \tag{3.52}$$

Thus, in this case we have two nontrivial components of the spin tensor $S^{23,0}$ and $S^{12,0}$. those define the projections of spin vector on X and Z axis, respectively. From (2.27) we write the chronometric invariant spin tensor

$$S^{23,0}_{\rm ch} = C_3^2 \sinh[2(\mathcal{A} + C_4)]e^{-\alpha}, \tag{3.53a}$$
$$S^{23,0}_{\rm ch} = C_3^2 e^{-\alpha} \tag{3.53b}$$

and the projections of the spin vector on X and Z axes are

$$S_1 = C_3^2 \int\limits_{-\infty}^{\infty} \sinh[2(\mathcal{A} + C_4)]e^{\alpha-\rho} dx, \tag{3.54a}$$

$$S_3 = C_3^2 \int\limits_{-\infty}^{\infty} e^{\alpha-\rho} dx. \tag{3.54b}$$

Note that the equation for α, therefore for P will be the same as in previous case (i.e., for S with $m = 0$) with all the conclusions made there. So we will not proceed further with this. We also note that for $F = K_\pm$ with $K_\pm = I \pm J$ for massless spinor field we obtain $K_\pm = K_0 e^{-2\alpha}$ and the conclusions made above will be remain valid.

IV. CONCLUSION

A nonlinear spinor field has been thoroughly studied within the scope of general relativity given by a plane-symmetric space-time. Energy density and the total energy of the linear spinor field are not bounded and the system does not possess real physical infinity, hence the configuration is not observable for an infinitely remote observer, since in this case

$$R = \int_{-\infty}^{\infty} \sqrt{g_{11}} dx = \int_{-\infty}^{\infty} e^{\alpha} dx = \frac{4C_0 H}{M^2} < \infty. \tag{4.1}$$

But introduction of nonliear spinor term into the system eliminates these shortcomings and we have the configuration with finite energy density and limited total energy which is also observable as in this case the system possesses real physical infinity. Thus we see, spinor field nonlinearity is crucial for the regular solutions with localized energy density.

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