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ON THE NATURAL GAUGE FIELDS OF MANIFOLDS

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The gauge symmetry inherent in the concept of manifold has been discussed. Within the scope of this symmetry the linear connection or displacement field can be considered as a natural gauge field on the manifold. The gauge-invariant equations for the displacement field have been derived. It has been shown that the energy–momentum tensor of this field conserves and hence the displacement field can be treated as one that transports energy and gravitates. To show the existence of the solutions of the field equations, we have derived the general form of the displacement field in Minkowski space–time which is invariant under rotation and space and time inversion. With this ansatz we found spherically-symmetric solutions of the equations in question.

Keywords: Symmetry, gauge field, displacement field, space–time theory

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1. Introduction

According to the modern standpoint, space–time theory is the one that possesses a mathematical representation whose elements are a smooth four-dimensional manifold $\mathcal{M}$ and geometric objects defined on this manifold. The geometry on the manifold is defined by metric and linear connection. In general, the linear connection is in no way related to the metric since these concepts define different geometric operations on the manifold $\mathcal{M}$. The metric on the manifold defines the length of a curve while the linear connection defines parallel transport (displacement) of vectors along arbitrary path on $\mathcal{M}$. It should be emphasized that soon after the creation of General Relativity, Eddington put forward the idea to derive all theories on the basis of parallel displacement only.[1] Here the metric and the linear connection as a totally independent geometric objects by structure will be considered as fundamental fields. It is our principal assumption. According to the fundamental idea of Einstein, metric corresponds to gravitational field while all other fields, being the source of gravitational one, carry energy. Hence and from the above assumption it follows that, like the electromagnetic field, the field of parallel displacement

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carries energy and appears to be the source of gravitational field, possessing geometric meaning. Thus, our aim is to derive natural equations for the field of parallel displacement and obtain the relevant conserving energy–momentum tensor, i.e. to show that within the framework of the canonical Einstein theory of gravity the linear connection can be considered on the same level with electromagnetic one.

2. Symmetry Group

There are two symmetry groups closely connected with the concept of manifold. One of them is a group of transformations of the manifold $\mathcal{M}$ itself, the manifold mapping group, and the other is a group of transformations acting in tangent vector spaces $T_p(\mathcal{M})$. The latter concept is clearly expounded in the treatise by Misner and Thorne and Wheeler.[2] The well-known manifold mapping group[3] is often called the group of general transformations of coordinates or the group of diffeomorphisms. The physical meaning of the manifold mapping group is that it is a group of symmetry of gravitational interactions in Einstein theory of gravity. A systematic and thorough consideration of the questions connected with space–time symmetry of General Relativity may be found in Ref. 3. We emphasize only that the diffeomorphism group is evidently the widest group of space–time symmetry.

Let the given vector field V^i undergo infinitesimal parallel displacement, then

$$dV^i + \Gamma^{\ i}_{jk} V^k \, dx^j = 0 \,, \tag{2.1}$$

where $\Gamma^{\ i}_{jk}$ are the components of linear connection. Vector fields form linear vector space L. The isomorphic mapping of the vector space L onto itself is defined by the tensor fields of type $(1,1)$. Let S^i_j be the components of a tensor field S of type $(1,1)$ that satisfies the condition $\det(S^i_j) \neq 0$ only. In this case, there exists a tensor field S^{-1} with components T^i_j such that $S^i_k T^k_j = \delta^i_j$. Now a tensor field S can be regarded as an isomorphism of L onto itself

$$\bar{V}^i(x) = S^i_j(x) V^j(x) \,. \tag{2.2}$$

Since there is no objective reason to distinguish vector fields $\bar{V}^i(x)$ and $V^j(x)$, we want to define the law of parallel displacement for the vector $\bar{V}^i$ induced by (2.1) and (2.2). It can be shown that if the vector V^i undergoes parallel displacement (2.1) then $\bar{V}^i$, defined by (2.2), undergoes parallel displacement

$$d\bar{V}^i + \bar{\Gamma}^{\ i}_{jk} \bar{V}^k \, dx^j = 0 \,, \tag{2.3}$$

where

$$\bar{\Gamma}^{\ i}_{jk} = S^i_l \Gamma^{\ l}_{jm} T^m_k + S^i_l \partial_j T^l_k \,, \tag{2.4}$$

and T^i_j are components of the field S^{-1} inverse to S. In what follows we will consider the transformation group (2.2) as the natural group of gauge transformation inherent in the manifold $\mathcal{M}$. The transformation (2.4) gives the realization of gauge group on the fields of parallel displacement and in that sense $\Gamma^{\ i}_{jk}$ are analogical to the potentials of electromagnetic fields. Now our aim is to find equation for Γ that is invariant under (2.4). From (2.4) it follows that if $\Gamma^{\,i}_{jk}$ are the components of linear connection, then $\bar{\Gamma}^{\,i}_{jk}$ are also the components of linear connection, i.e. under coordinate transformation $\bar{\Gamma}$ transforms in accordance with the same well-known laws as does Γ itself.[3]

3. Gauge-Invariant Equations

As it is noted above, the diffeomorphism group is responsible for gravitational interactions, and thus, the gauge group under consideration is a symmetry group of new interactions. To simplify computations and to write equations in a symmetrical and manifestly gauge-invariant form, we introduce the notion of the gauge derivative. We will say that a tensor field T of type (m, n) is of gauge type (p, q) if under the transformations of gauge group there is the correspondence

$$T \Rightarrow \bar{T} = \underbrace{S \cdots S}_{p} T \underbrace{S^{-1} \cdots S^{-1}}_{q},$$

where

$$0 \leq p \leq m \qquad \text{and} \qquad 0 \leq q \leq n\,.$$

The Einstein potentials g_{ij} being a tensor field of type $(0, 2)$ is to be assigned the gauge type $(0, 0)$ because the Einstein equations are not invariant with respect to the transformations $\bar{g}_{ij} = g_{kl} T^k_i T^l_j$. Let the vector field V^i has a gauge type $(1, 0)$. We define gauge derivative as

$$D_j V^k = \partial_j V^k + \Gamma^{\,k}_{jl} V^l\,.$$

Now, if the equality

$$\bar{D}_j \bar{V}^k = \partial_j \bar{V}^k + \bar{\Gamma}^{\,k}_{jl} \bar{V}^l$$

holds, then from (2.2) and (2.4), it follows that

$$\bar{D}_i \bar{V}^j = S^j_k D_i V^k\,.$$

Since

$$D_i D_j V^k = \partial_i (D_j V^k) + \Gamma^{\,k}_{il} (D_j V^l)\,,$$

$D_i D_j V^k$ is not a tensor field, nevertheless, the commutator of gauge derivatives are tensor fields, as

$$[D_i, D_j] V^k = B_{ijl}^{\;\;\;\,k} V^l\,, \tag{3.1}$$

where

$$B_{ijl}^{\;\;\;\,k} = \partial_i \Gamma^{\,k}_{jl} - \partial_j \Gamma^{\,k}_{il} + \Gamma^{\,k}_{im} \Gamma^{\,m}_{jl} - \Gamma^{\,k}_{jm} \Gamma^{\,m}_{il}\,, \tag{3.2}$$

is the Riemann tensor of curvature of connection Γ. Note that $B_{ijl}^{\ \ \ k}$ is a tensor field of type (1, 3) and gauge type (1, 1). In what follows, for brevity, we use matrix notation, assuming that

$$\Gamma_i = (\Gamma_{ij}^{\ k}), \qquad B_{ij} = (B_{ijl}^{\ \ \ k}), \qquad \mathrm{Tr}\, B_{ij} = B_{ijk}^{\ \ \ k}, \qquad \Gamma_i \Gamma_j = \Gamma_{im}^{\ k} \Gamma_{jl}^{\ m} .$$

In matrix notation

$$B_{ij} = \partial_i \Gamma_j - \partial_j \Gamma_j + [\Gamma_i, \Gamma_j] \,, \tag{3.3}$$

$$\bar{\Gamma}_i = S\Gamma_i S^{-1} + S\partial_i S^{-1} \,, \tag{3.4}$$

$$\bar{B}_{ij} = SB_{ij}S^{-1} \,. \tag{3.5}$$

It is obvious from (3.3)–(3.5) that, like $F_{ij} = \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i$, B_{ij} is strength tensor. The generally covariant and gauge-invariant Lagrangian for gauge field $\Gamma_{ij}^{\ k}$ (displacement field) has the form

$$\mathcal{L} = -\frac{1}{4} \mathrm{Tr}(B^{ij} B_{ij}) \,, \tag{3.6}$$

where

$$B^{ij} = g^{ik} g^{jl} B_{kl}$$

and g^{ij} is a tensor field inverse to g_{ij} such that $g_{jk} g^{ik} = \delta_j^i$.

Varying (3.6) with respect to Γ, we obtain the following system of second-order differential equations for the displacement field

$$\frac{1}{\sqrt{-g}} D_i (\sqrt{-g} B^{ij}) = 0 \,. \tag{3.7}$$

In fact, if $\delta\Gamma_i$ is variation, then

$$\delta B_{ij} = D_i \delta\Gamma_j - D_j \delta\Gamma_i \,.$$

Hence it follows that

$$\delta\mathcal{L} = -\mathrm{Tr}(B^{ij} D_i \delta\Gamma_j) = -\partial_i \mathcal{J}^i + \mathrm{Tr}((D_i B^{ij}) \delta\Gamma_j) \,,$$

where $\mathcal{J}^i = \mathrm{Tr}(B^{ij} \delta\Gamma_j)$. Since

$$\partial_i \mathcal{J}^i = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} \mathcal{J}^i) - \left(\frac{\partial_i \sqrt{-g}}{\sqrt{-g}} \right) \mathcal{J}^i \,,$$

$$\delta\mathcal{L} = -\frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} \mathcal{J}^i) + \mathrm{Tr}\left(\frac{1}{\sqrt{-g}} D_i (\sqrt{-g} B^{ij}) \delta\Gamma_j \right) .$$

Q.E.D.

Varying the action

$$\mathcal{A} = \int dx^4 \, \mathcal{L} \sqrt{-g}$$

with respect to the metric g_{ij} we obtain gauge-invariant energy–momentum metric tensor for displacement field Γ

$$T^{ij} = \mathrm{Tr}(B^{ik} B^{j}{}_{k}) - \frac{1}{4} g^{ij} (B_{kl} B^{kl}) \tag{3.8}$$

which on the solutions of Eq. (3.7) satisfies the local law of energy conservation

$$T^{ij}{}_{;j} = 0 \,. \tag{3.9}$$

Here semicolon denotes the covariant derivative with respect to the Levi–Cività connection belonging to the metric g_{ij}

$$\{{}^{\,i}_{jk}\} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}) \,. \tag{3.10}$$

In view of its significance, we underline few details of the proof of the relation (3.9). We have

$$T^{ij}{}_{;j} = \partial_j T^{ij} + \{{}^{\,j}_{jk}\} T^{ik} + \{{}^{\,i}_{jk}\} T^{jk} = \frac{1}{\sqrt{-g}} \partial_j (\sqrt{-g} T^{ij}) + \{{}^{\,i}_{jk}\} T^{jk} \,. \tag{3.11}$$

Since, according to (3.10)

$$\partial_i g^{jk} = -\{{}^{\,j}_{il}\} g^{kl} - \{{}^{\,k}_{il}\} g^{jl} \,, \tag{3.12}$$

it can be shown that

$$\begin{aligned} \mathrm{Tr}(B^{jk} D_j B^{i}{}_{k}) = {} & -\{{}^{\,i}_{jk}\} \,\mathrm{Tr}(B^{jl} B^{k}_{\ l}) - \{{}^{\,l}_{jk}\} g^{ik} \,\mathrm{Tr}(B^{jm} B_{lm}) \\ & + \frac{1}{2} g^{ik} \,\mathrm{Tr}(B^{jl} (D_j B_{kl} - D_l B_{kj})) \,, \end{aligned} \tag{3.13}$$

$$\mathrm{Tr}(B_{kl} D_j B^{kl}) = \mathrm{Tr}(B^{kl} D_j B_{kl}) - 4 \{{}^{\,k}_{jl}\} \,\mathrm{Tr}(B_{km} B^{lm}) \,. \tag{3.14}$$

From (3.11), (3.13) and (3.14), it follows that

$$T^{ij}{}_{;j} = \mathrm{Tr}\left(\frac{1}{\sqrt{-g}} D_j (\sqrt{-g} B^{jk}) B^{i}{}_{k} \right) + \frac{1}{2} g^{ik} \,\mathrm{Tr}(B^{jl} (D_j B_{kl} + D_k B_{lj} + D_l B_{jk})) \,.$$

Since the equation

$$D_j B_{kl} + D_k B_{lj} + D_l B_{jk} = 0$$

is fulfilled identically, the local law of energy–momentum conservation (3.9) is also fulfilled for the case in question.

Our conclusion is that Eq. (3.7) and the Einstein equations

$$R_{ij} - \frac{1}{2} g_{ij} R = \kappa T_{ij} \,, \tag{3.15}$$

with the right-hand side given by the expression (3.8), form a consistent system of partial differential equations which is invariant under gauge transformations as well as under the transformations of diffeomorphism group. Now we have proven that the displacement field Γ is really the origin of gravitational field within the scope of the given gauge approach.

4. Spherical-Symmetrical Gauge Potentials

As the first step to investigate Eq. (3.7), it is very important to show that they have nontrivial solutions. In doing this we show that Eq. (3.7) possesses spherically symmetric solutions.

The general theory of space–time symmetry within the scope of theory of gauge fields has been developed in Refs. 6 and 7. We apply the results obtained there to our particular case. Note that the spherically symmetric solutions of SU(2) Yang–Mills equations were first derived by Ikeda and Miyachi[8] and for SU(3) by Loos.[9]

In this section we consider Minkowski space–time with the metric in spherical system of coordinates that is most convenient under the consideration of SO(3) symmetry:

$$ds^2 = dt^2 - dr^2 - r^2\, d\vartheta^2 - r^2 \sin^2 \vartheta\, d\varphi^2\,, \tag{4.1}$$

where c has been taken to be unity.

First of all we would like to find gauge potentials which are invariant under the displacement along the time axis $t \to t + a$, i.e.

$$\Gamma^i_{jk}(x^0, x^1, x^2, x^3) = \Gamma^i_{jk}(x^0 + a, x^1, x^2, x^3)\,. \tag{4.2}$$

From (4.2) it follows that all Γ's are independent of t. Now we shall look for SO(3) invariant gauge potentials. Generators of SO(3) group in spherical coordinates have the form[5]

$$X_1 = \sin\varphi \frac{\partial}{\partial\vartheta} + \cot\vartheta\, \cos\varphi \frac{\partial}{\partial\varphi}\,, \tag{4.3a}$$

$$X_2 = -\cos\varphi \frac{\partial}{\partial\vartheta} + \cot\vartheta\, \sin\varphi \frac{\partial}{\partial\varphi}\,, \tag{4.3b}$$

$$X_3 = -\frac{\partial}{\partial\varphi}\,, \tag{4.3c}$$

and hence, the problem is to find solutions of the equations $L_X\Gamma = 0$ when Lie derivatives are taken along the vector fields of SO(3) group. The equation $L_{X_a}\Gamma = 0$, $a = 1, 2, 3$ can be written in the following matrix representation:

$$L_{X_a}\Gamma_j = V^\ell_{(a)}\partial_\ell\Gamma_j + [\Gamma_j, A_{(a)}] + \Gamma_\ell A^\ell_{(a)j} + \partial_j A_{(a)} = 0\,. \tag{4.4}$$

Here, $V^\ell_{(a)}$ are defined from $X_{(a)} = V^\ell_{(a)}\partial_\ell$ as

$$V^\ell_{(1)} = (0,\, 0, \sin\varphi,\, \cot\vartheta\cos\varphi)\,,$$

$$V^\ell_{(2)} = (0,\, 0,\, -\cos\varphi,\, \cot\vartheta\sin\varphi)\,,$$

$$V^\ell_{(3)} = (0, 0, 0, -1)\,.$$

The matrices $A_{(a)}$ here take the forms

$$A_{(1)} = A^i_{(1)j} = \partial_j V^i_{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{A}_{(1)} \end{pmatrix}, \quad \tilde{A}_{(1)} = \begin{pmatrix} 0 & \cos\varphi \\ -\cos\varphi/\sin^2\vartheta & -\cot\vartheta\sin\varphi \end{pmatrix},$$

$$A_{(2)} = A^i_{(2)j} = \partial_j V^i_{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{A}_{(2)} \end{pmatrix}, \quad \tilde{A}_{(2)} = \begin{pmatrix} 0 & \sin\varphi \\ -\sin\varphi/\sin^2\vartheta & -\cot\vartheta\cos\varphi \end{pmatrix},$$

$$A_{(3)} = A^i_{(3)j} = \partial_j V^i_3 = 0\,.$$

Let us write the equations $L_{X_{(a)}}\Gamma_j = 0$ explicitly. The equations $L_{X_{(1)}}\Gamma_j = 0$ can be written as follows:

$$\sin\varphi\frac{\partial\Gamma_0}{\partial\vartheta} + \cot\vartheta\cos\varphi\frac{\partial\Gamma_0}{\partial\varphi} + [\Gamma_0, A_{(1)}] = 0\,, \tag{4.5a}$$

$$\sin\varphi\frac{\partial\Gamma_1}{\partial\vartheta} + \cot\vartheta\cos\varphi\frac{\partial\Gamma_1}{\partial\varphi} + [\Gamma_1, A_{(1)}] = 0\,, \tag{4.5b}$$

$$\sin\varphi\frac{\partial\Gamma_2}{\partial\vartheta} + \cot\vartheta\cos\varphi\frac{\partial\Gamma_2}{\partial\varphi} + [\Gamma_2, A_{(1)}] - \frac{\cos\varphi}{\sin^2\vartheta}\Gamma_3 + \frac{\partial A_{(1)}}{\partial\vartheta} = 0\,, \tag{4.5c}$$

$$\begin{aligned} &\sin\varphi\frac{\partial\Gamma_3}{\partial\vartheta} + \cot\vartheta\cos\varphi\frac{\partial\Gamma_3}{\partial\varphi} + [\Gamma_3, A_{(1)}] + \cos\varphi\Gamma_2 \\ &\quad - \cot\vartheta\sin\varphi\Gamma_3 + \frac{\partial A_{(1)}}{\partial\varphi} = 0\,, \end{aligned} \tag{4.5d}$$

while the equations $L_{X_{(2)}}\Gamma_j = 0$ read

$$-\cos\varphi\frac{\partial\Gamma_0}{\partial\vartheta} + \cot\vartheta\sin\varphi\frac{\partial\Gamma_0}{\partial\varphi} + [\Gamma_0, A_{(2)}] = 0\,, \tag{4.6a}$$

$$-\cos\varphi\frac{\partial\Gamma_1}{\partial\vartheta} + \cot\vartheta\sin\varphi\frac{\partial\Gamma_1}{\partial\varphi} + [\Gamma_1, A_{(2)}] = 0\,, \tag{4.6b}$$

$$-\cos\varphi\frac{\partial\Gamma_2}{\partial\vartheta} + \cot\vartheta\sin\varphi\frac{\partial\Gamma_2}{\partial\varphi} + [\Gamma_2, A_{(2)}] - \frac{\sin\varphi}{\sin^2\vartheta}\Gamma_3 + \frac{\partial A_{(2)}}{\partial\vartheta} = 0\,, \tag{4.6c}$$

$$\begin{aligned} &-\cos\varphi\frac{\partial\Gamma_3}{\partial\vartheta} + \cot\vartheta\,\sin\varphi\frac{\partial\Gamma_3}{\partial\varphi} + [\Gamma_3, A_{(2)}] + \sin\varphi\Gamma_2 \\ &\quad + \cot\vartheta\cos\varphi\Gamma_3 + \frac{\partial A_{(2)}}{\partial\varphi} = 0\,. \end{aligned} \tag{4.6d}$$

Finally for $L_{X_{(3)}}\Gamma_j = 0$ we obtain

$$\frac{\partial\Gamma_j}{\partial\varphi} = 0\,. \tag{4.7}$$

Here assume that Γ_j are taken in the form

$$\Gamma_j = \begin{pmatrix} \Gamma^0_{j0} & \Gamma^0_{j1} & \Gamma^0_{j2} & \Gamma^0_{j3} \\ \Gamma^1_{j0} & \Gamma^1_{j1} & \Gamma^1_{j2} & \Gamma^1_{j3} \\ \Gamma^2_{j0} & \Gamma^2_{j1} & \Gamma^2_{j2} & \Gamma^2_{j3} \\ \Gamma^3_{j0} & \Gamma^3_{j1} & \Gamma^3_{j2} & \Gamma^3_{j3} \end{pmatrix}, \tag{4.8}$$

where the upper indices enumerate the rows. From Eq. (4.7) it follows that the Γ_j's are independent of φ. Taking into account that the Γ_j's are independent of t and φ we finally combine the foregoing Eqs. (4.5) and (4.6) in the form

$$[\Gamma_0, C] = 0, \qquad \frac{\partial \Gamma_0}{\partial \vartheta} + [\Gamma_0, D] = 0\,, \tag{4.9}$$

$$[\Gamma_1, C] = 0, \qquad \frac{\partial \Gamma_1}{\partial \vartheta} + [\Gamma_1, D] = 0\,, \tag{4.10}$$

$$[\Gamma_2, C] - \frac{1}{\sin^2 \vartheta}\Gamma_3 + \frac{\partial C}{\partial \vartheta} = 0, \qquad \frac{\partial \Gamma_2}{\partial \vartheta} + [\Gamma_2, D] + \frac{\partial D}{\partial \vartheta} = 0\,, \tag{4.11}$$

$$[\Gamma_3, C] + \Gamma_2 + D = 0, \qquad \frac{\partial \Gamma_3}{\partial \vartheta} + [\Gamma_3, D] - \cot\vartheta \Gamma_3 - C = 0\,, \tag{4.12}$$

where we define

$$C = \cos\varphi A_{(1)} + \sin\varphi A_{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{C} \end{pmatrix}, \qquad \tilde{C} = \begin{pmatrix} 0 & 1 \\ -1/\sin^2\vartheta & 0 \end{pmatrix},$$

$$D = \sin\varphi A_{(1)} - \cos\varphi A_{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D} \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} 0 & 0 \\ 0 & -\cot\vartheta \end{pmatrix}.$$

Solving Eqs. (4.9)–(4.12), we find Γ_j's which are independent of t and φ

$$\begin{aligned}
\Gamma_0 &= \begin{pmatrix} a & \alpha & 0 & 0 \\ \beta & b & 0 & 0 \\ 0 & 0 & c & -d\sin\vartheta \\ 0 & 0 & d/\sin\vartheta & c \end{pmatrix}, \\
\Gamma_1 &= \begin{pmatrix} \gamma & h & 0 & 0 \\ k & \delta & 0 & 0 \\ 0 & 0 & \mu & -\nu\sin\vartheta \\ 0 & 0 & \nu/\sin\vartheta & \mu \end{pmatrix}, \\
\Gamma_2 &= \begin{pmatrix} 0 & 0 & p & q\sin\vartheta \\ 0 & 0 & \sigma & \tau\sin\vartheta \\ m & \lambda & 0 & 0 \\ n/\sin\vartheta & \varepsilon/\sin\vartheta & 0 & \cot\vartheta \end{pmatrix}, \\
\Gamma_3 &= \begin{pmatrix} 0 & 0 & -q\sin\vartheta & p\sin^2\vartheta \\ 0 & 0 & -\tau\sin\vartheta & \sigma\sin^2\vartheta \\ -n\sin\vartheta & -\varepsilon\sin\vartheta & 0 & -\sin\vartheta\cos\vartheta \\ m & \lambda & \cot\vartheta & 0 \end{pmatrix}.
\end{aligned} \tag{4.13}$$

Thus we found the general spherically symmetric ansatz for displacement field Γ. All the unknown functions in (4.13) are the arbitrary functions of r only.

Now the problem is to find these functions as the solutions of Eq. (3.7). Taking into account that $\partial_t \Gamma_j = 0$ and $\partial_\varphi \Gamma_j = 0$, from (3.2) and (3.3) for the nontrivial components of the Riemann tensor we find

$$B_{10} = \frac{\partial \Gamma_0}{\partial r} + [\Gamma_1, \Gamma_0]\,, \tag{4.14a}$$

$$B_{20} = \frac{\partial \Gamma_0}{\partial \vartheta} + [\Gamma_2, \Gamma_0]\,, \tag{4.14b}$$

$$B_{30} = [\Gamma_3, \Gamma_0]\,, \tag{4.14c}$$

$$B_{12} = \frac{\partial \Gamma_2}{\partial r} - \frac{\partial \Gamma_1}{\partial \vartheta} + [\Gamma_1, \Gamma_2]\,, \tag{4.14d}$$

$$B_{13} = \frac{\partial \Gamma_3}{\partial r} + [\Gamma_1, \Gamma_3]\,, \tag{4.14e}$$

$$B_{23} = \frac{\partial \Gamma_3}{\partial \vartheta} + [\Gamma_2, \Gamma_3]\,. \tag{4.14f}$$

Putting (4.13) into (4.14) one can find the nontrivial components of the Riemann tensor B_{ij}. But we shall not do that since, for further simplification of our problem we demand the Γ_j's to be invariant under time inversion, i.e. under

$$t \to t' = -t, \quad r \to r' = r, \qquad \vartheta \to \vartheta' = \vartheta, \qquad \varphi \to \varphi' = \varphi$$

the Γ_j's should remain unaltered. Let us explain from the general point of view what does it mean. Let the transformation ϕ on the manifold $\mathcal{M}$ maps coordinate patch U onto itself. The transformation ϕ can be represented by smooth functions in U

$$\phi : x^i \Longrightarrow \phi^i(x); \quad \phi^{-1} : x^i \Longrightarrow f^i(x); \quad \phi^i(f(x)) = x^i\,.$$

Under ϕ, Γ transforms as follows:

$$\tilde{\Gamma}^{\,s}_{jk}(x) = \phi^s_l(f(x))\Gamma^{\,l}_{mn}(f(x))f^m_j(x)f^n_k(x) + \phi^s_l(f(x))\partial_j f^l_k(x)\,, \tag{4.15}$$

where $f^s_l = \partial_l f^s(x), \phi^s_l = \partial_l \phi^s(x)$. It is said that the field with components $\Gamma^{\,i}_{jk}$ is invariant with respect to the transformation ϕ if

$$\tilde{\Gamma}^{\,s}_{jk}(x) = \phi^s_l(f(x))\Gamma^{\,l}_{mn}(f(x))f^m_j(x)f^n_k(x) + \phi^s_l(f(x))\partial_j f^l_k(x) = \Gamma^{\,s}_{jk}\,. \tag{4.16}$$

At infinitesimal ϕ, when $f^i(x) = x^i + v^i\epsilon$, from (4.16) it follows that $L_X\Gamma^{\,i}_{jl} = 0$ where $X = v^i \frac{\partial}{\partial x^i}$.

In case of time inversion $f^0(x) = -t, f^1(x) = r, f^2(x) = \vartheta$ and $f^3(x) = \varphi$, hence

$$F = f^l_k(x) = \frac{\partial f^l(x)}{\partial x^k} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Taking into account that $\partial_j f^l_k(x) = 0$, multiplying (4.15) by $f^i_s(x)$ from the left after a little manipulation we find the transformation law for Γ_j's

$$F\Gamma_j(x) = [f^m_j \Gamma_m(f(x))]F\,, \tag{4.17}$$

or more explicitly

$$F\Gamma_0(x) = -\Gamma_0(f(x))F, \qquad F\Gamma_\mu(x) = \Gamma_\mu(f(x))F, \qquad \mu = 1, 2, 3\,. \tag{4.18}$$

From (4.18) we find

$$\Gamma^\mu_{0\nu} = 0, \quad \Gamma^0_{00} = 0, \quad \Gamma^0_{\mu 1} = \Gamma^0_{\mu 2} = \Gamma^0_{\mu 3} = 0, \quad \Gamma^1_{\mu 0} = \Gamma^2_{\mu 0} = \Gamma^3_{\mu 0} = 0, \quad \mu,\, \nu = 1, 2, 3\,. \tag{4.19}$$

Thus the Γ_j's are spherically symmetric and invariant under time inversion:

$$\Gamma_0 = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \mu & -\nu \sin\vartheta \\ 0 & 0 & \nu/\sin\vartheta & \mu \end{pmatrix},$$
$$\Gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & \tau \sin\vartheta \\ 0 & \lambda & 0 & 0 \\ 0 & \varepsilon/\sin\vartheta & 0 & \cot\vartheta \end{pmatrix}, \tag{4.20}$$
$$\Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau\sin\vartheta & \sigma\sin^2\vartheta \\ 0 & -\varepsilon\sin\vartheta & 0 & -\sin\vartheta\cos\vartheta \\ 0 & \lambda & \cot\vartheta & 0 \end{pmatrix}.$$

Now putting (4.20) into (4.14) we obtain the following nontrivial components of the Riemann tensor

$$B_{10} = \begin{pmatrix} 0 & \bar\alpha & 0 & 0 \\ \bar\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_{20} = \begin{pmatrix} 0 & 0 & -\alpha\sigma & -\alpha\tau\sin\vartheta \\ 0 & 0 & 0 & 0 \\ \lambda\beta & 0 & 0 & 0 \\ \varepsilon\beta/\sin\vartheta & 0 & 0 & 0 \end{pmatrix},$$
$$B_{30} = \begin{pmatrix} 0 & 0 & -\alpha\tau\sin\vartheta & -\alpha\sigma\sin^2\vartheta \\ 0 & 0 & 0 & 0 \\ -\varepsilon\beta\sin\vartheta & 0 & 0 & 0 \\ \lambda\beta & 0 & 0 & 0 \end{pmatrix},$$
$$B_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \bar\sigma & \bar\tau\sin\vartheta \\ 0 & \bar\lambda & 0 & 0 \\ 0 & \bar\varepsilon/\sin\vartheta & 0 & 0 \end{pmatrix}, \quad B_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \bar\tau\sin\vartheta & -\bar\sigma\sin^2\vartheta \\ 0 & \bar\sigma\sin\vartheta & 0 & 0 \\ 0 & -\bar\lambda & 0 & 0 \end{pmatrix}, \tag{4.21}$$
$$B_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2A\sin\vartheta & 0 & 0 \\ 0 & 0 & A\sin\vartheta & B\sin^2\vartheta \\ 0 & 0 & -B & A\sin\vartheta \end{pmatrix},$$

where we define

$$\bar{\alpha} := \alpha' - \alpha(\delta - \gamma), \qquad \bar{\beta} := \beta' + \beta(\delta - \gamma)\,,$$

$$\bar{\sigma} := \sigma' + \sigma(\delta - \mu) - \tau\nu, \qquad \bar{\tau} := \tau' + \tau(\delta - \mu) + \sigma\nu\,,$$

$$\bar{\lambda} := \lambda' - \lambda(\delta - \mu) - \varepsilon\nu, \qquad \bar{\varepsilon} := \varepsilon' - \varepsilon(\delta - \mu) + \lambda\nu\,,$$

$$A := \varepsilon\sigma - \tau\lambda, \qquad B := \varepsilon\tau + \sigma\lambda + 1\,.$$

From (3.8) we obtain energy-density for the displacement field $\Gamma^{\,i}_{jk}$

$$\begin{aligned} T_{00} = {} & -\bar{\alpha}\bar{\beta} + \frac{2}{r^2}\alpha\beta(\sigma\lambda + \tau\varepsilon) - \frac{2}{r^2}(\bar{\sigma}\bar{\lambda} + \bar{\tau}\bar{\varepsilon}) \\ & - \frac{2}{r^4}(\lambda\tau - \varepsilon\sigma)^2 - \frac{1}{r^4}[(\varepsilon\sigma - \tau\lambda)^2 - (\varepsilon\tau + \sigma\lambda + 1)^2]\,. \end{aligned} \tag{4.22}$$

Once the Riemann tensor is defined, we immediately undertake to write the equations for the functions under consideration. To this end we invoke Eq. (3.7) that can be rewritten in the form

$$\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}B^{ij}) + [\Gamma_i, B^{ij}] = 0\,. \tag{4.23}$$

Here $B^{ij} = B_{pq}g^{ip}g^{jq}$, $\sqrt{-g} = r^2\sin\vartheta$ and $g_{ij} = \mathrm{diag}(1, -1, -r^2, -r^2\sin\vartheta)$. Inserting (4.20) and (4.21) into (4.23) we obtain

$$\bar{\alpha}' + \frac{2}{r}\bar{\alpha} + \frac{2}{r^2}(B - 1)\alpha = 0\,, \tag{4.24a}$$

$$\bar{\beta}' + \frac{2}{r}\bar{\beta} + \frac{2}{r^2}(B - 1)\beta = 0\,, \tag{4.24b}$$

$$\alpha\bar{\beta} - \beta\bar{\alpha} = 0\,, \tag{4.24c}$$

$$\bar{\sigma}\lambda - \bar{\lambda}\sigma + \bar{\tau}\varepsilon - \bar{\varepsilon}\tau = 0\,, \tag{4.24d}$$

$$\bar{\lambda}\tau - \bar{\tau}\lambda + \bar{\sigma}\varepsilon - \bar{\varepsilon}\sigma = 0\,, \tag{4.24e}$$

$$\bar{\sigma}' - (\mu - \delta)\bar{\sigma} - \alpha\beta\sigma - \bar{\tau}\nu + \frac{1}{r^2}(B\sigma + 3A\tau) = 0\,, \tag{4.24f}$$

$$\bar{\tau}' - (\mu - \delta)\bar{\tau} - \alpha\beta\tau + \bar{\sigma}\nu + \frac{1}{r^2}(B\tau - 3A\sigma) = 0\,, \tag{4.24g}$$

$$\bar{\lambda}' + (\mu - \gamma)\bar{\lambda} - \alpha\beta\lambda - \bar{\varepsilon}\nu + \frac{1}{r^2}(B\lambda - 3A\varepsilon) = 0\,, \tag{4.24h}$$

$$\bar{\varepsilon}' + (\mu - \gamma)\bar{\varepsilon} - \alpha\beta\varepsilon + \bar{\lambda}\nu + \frac{1}{r^2}(B\varepsilon + 3A\lambda) = 0\,. \tag{4.24i}$$

The system (4) contains ten unknown functions, but there is no equation for γ, δ, μ, ν which determine Γ_1. Let us demand the Γ_j's be invariant under space inversion, i.e. under

$$t \to t' = t, \qquad r \to r' = r, \qquad \vartheta \to \vartheta' = \pi - \vartheta, \qquad \varphi \to \varphi' = \varphi$$

the Γ_j's should remain unaltered. In this case F in (4.17) reads

$$F = f_k^l(x) = \frac{\partial f^l(x)}{\partial x^k} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence (4.17) explicitly reads

$$F\Gamma_2(x) = -\Gamma_2(f(x))F, \qquad F\Gamma_i(x) = \Gamma_i(f(x))F, \qquad i = 0, 1, 3. \tag{4.25}$$

From (4.25) we find $\nu = 0, \tau = 0, \varepsilon = 0$. Thus the Γ_j's those are spherically symmetric and invariant under time and space inversion take the form

$$\begin{aligned} \Gamma_0 &= \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_1 = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \\ \Gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \cot\vartheta \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma\sin^2\vartheta \\ 0 & 0 & 0 & -\sin\vartheta\cos\vartheta \\ 0 & \lambda & \cot\vartheta & 0 \end{pmatrix}. \end{aligned} \tag{4.26}$$

We again see that $\Gamma_1 \neq 0$. In view of this let us consider gauge transformations which leave the equation $L_X\Gamma = 0$ invariant, i.e. find transformations S such that $L_X\Gamma = 0$ implies $L_X\bar{\Gamma} = 0$, where $\bar{\Gamma}$ is given by (3.4)

$$\bar{\Gamma}_i = S\Gamma_i S^{-1} + S\partial_i S^{-1}.$$

The natural choice for the $L_X\Gamma = 0$ to be gauge-invariant is to put

$$L_X S = 0 \tag{4.27}$$

or explicitly

$$\sin\varphi\frac{\partial S}{\partial\vartheta} + \cot\vartheta\cos\varphi\frac{\partial S}{\partial\varphi} + [S, A_{(1)}] = 0, \tag{4.28a}$$

$$-\cos\varphi\frac{\partial S}{\partial\vartheta} + \cot\vartheta\sin\varphi\frac{\partial S}{\partial\varphi} + [S, A_{(2)}] = 0, \tag{4.28b}$$

$$\frac{\partial S}{\partial\varphi} = 0. \tag{4.28c}$$

In account of (4.28c) we combine (4.28a) and (4.28b) together to get the equations for determining S:

$$\frac{\partial S}{\partial\vartheta} - [S, D] = 0, \tag{4.29a}$$

$$[S, C] = 0. \tag{4.29b}$$

General solution of (4.29) takes the form

$$S = \begin{pmatrix} \tilde{a} & \tilde{b} & 0 & 0 \\ \tilde{c} & \tilde{d} & 0 & 0 \\ 0 & 0 & \tilde{e} & -\tilde{f}\sin\vartheta \\ 0 & 0 & \tilde{f}/\sin\vartheta & \tilde{e} \end{pmatrix}, \tag{4.30}$$

with $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}$ being the functions of r only. Now, our assumption of invariance under space and time inversion leads to the functions $\tilde{b}, \tilde{c}, \tilde{f}$ to be trivial. Hence we obtain the following expression for S:

$$S = \begin{pmatrix} \tilde{a} & 0 & 0 & 0 \\ 0 & \tilde{d} & 0 & 0 \\ 0 & 0 & \tilde{e} & 0 \\ 0 & 0 & 0 & \tilde{e} \end{pmatrix}. \tag{4.31}$$

Let us now use the gauge arbitrariness. In doing so we demand $\bar{\Gamma}_1$ to be zero. Then from (3.4), i.e.

$$\bar{\Gamma}_i = S\Gamma_i S^{-1} + S\partial_i S^{-1},$$

we obtain equation for fixing gauge

$$\frac{\partial S}{\partial r} = S\Gamma_1, \tag{4.32}$$

that yields the following results

$$\tilde{a} = \exp\left[\int \gamma\, dr\right], \qquad \tilde{d} = \exp\left[\int \delta\, dr\right] \tilde{e} = \exp\left[\int \mu\, dr\right]. \tag{4.33}$$

Thus, without loss of generality we can put $\Gamma_1 = 0$. Now the system (4.24) reduces to

$$\alpha'' + \frac{2}{r}\alpha' + \frac{2}{r^2}\sigma\lambda\alpha = 0, \tag{4.34a}$$

$$\beta'' + \frac{2}{r}\beta' + \frac{2}{r^2}\sigma\lambda\beta = 0, \tag{4.34b}$$

$$\alpha\beta' - \beta\alpha' = 0, \tag{4.34c}$$

$$\sigma'' - \alpha\beta\sigma + \frac{1}{r^2}(\sigma\lambda + 1)\sigma = 0, \tag{4.34d}$$

$$\lambda'' - \alpha\beta\lambda + \frac{1}{r^2}(\sigma\lambda + 1)\lambda = 0, \tag{4.34e}$$

$$\lambda\sigma' - \sigma\lambda' = 0. \tag{4.34f}$$

From (4.34c) and (4.34f) follow $\beta = c_0\alpha$ and $\lambda = d_0\sigma$, where c_0 and d_0 are some arbitrary constants. In this case from (4.22) we find

$$T_{00} = -c_0\alpha^{pr2} + \frac{2}{r^2}c_0 d_0\alpha^2\sigma^2 - \frac{2}{r^2}d_0\sigma'^2 + \frac{1}{r^4}(d_0\sigma^2 + 1)^2. \tag{4.35}$$

It is obvious from (4.35) that for the energy to be positive definite one should simply imply the constants c_0 and d_0 to be negative, i.e. $c_0 < 0$ and $d_0 < 0$.

In spherical coordinates the functions $\alpha, \beta, \sigma, \lambda$ and the constant d_0 have the following dimensions: $[\alpha] = L^{-1}, [\beta] = L^{-1}, [\sigma] = L^{-1}, [\lambda] = L, [d_0] = L^2$. The constant c_0 is dimensionless.

It is obvious that if the system (4) possesses nontrivial solutions, so does the system (3.7). One of the special solutions is $\alpha = \alpha_0/r, \beta = \beta_0/r, \lambda = 0$ and $\sigma = 0$.

Since the constant d_0 is not dimensionless, let us consider the case when $d_0 = 0$. In other words we assume the function λ to be zero. Under this condition from (4.34) we find $\alpha = \alpha_0/r$ and $\beta = \beta_0/r$. For σ we obtain the equation

$$r^2\sigma'' + (1 - \alpha_0\beta_0)\sigma = 0\,. \tag{4.36}$$

Introducing a dimensionless parameter $\varrho = r/l$, where l is a constant such that $[l] = L$, we rewrite Eq. (4.36)

$$\varrho^2 \frac{\partial^2\sigma}{\partial\varrho^2} + (1 - \alpha_0\beta_0)\sigma = 0\,. \tag{4.37}$$

Defining $b^2 = (1 - \alpha_0\beta_0)^2 - 1/4$, we find the following expressions for σ:

$$\frac{\sigma}{\sqrt{\varrho}} = \begin{cases} C_1\cos(b\ln\varrho) + C_2\sin(b\ln\varrho), & b^2 > 0 \\ C_1\varrho^b + C_2\varrho^{-b}, & b^2 < 0 \\ C_1 + C_2\ln\varrho, & b^2 = 0 \end{cases} \tag{4.38}$$

where the constants C_1 and C_2 have the dimension of length. Thus the system (4.34) possesses solution and so does the system (3.7).

5. Conclusion

Summarizing the results obtained we once again emphasize that within the framework of gauge symmetry inherent in the concept of manifold it is natural to consider the linear connection as a gauge field. Under the gauge symmetry condition it is impossible to demand the condition $\Gamma^k_{ij} = \Gamma^k_{ji}$ to be fulfilled, since it is not gauge-invariant. It is shown that the conserving energy–momentum tensor exists for the displacement field and hence, this field can be treated within the scope of GR as a material one with deep geometrical meaning.

To show the similarity of the classical displacement field with the electromagnetic one and to prove the existence of nontrivial solutions we have found the static spherically-symmetric ansatz. We have also shown that its insertion into Eq. (3.7) allows one to obtain the corresponding solutions.

Our conclusion is that together with the known long-range interactions there can exist new type of long-range interactions defined by displacement field that was the subject of our investigation.

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