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BIANCHI TYPE I UNIVERSE WITH VISCOUS FLUID

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We study the evolution of a homogeneous, anisotropic Universe given by a Bianchi type-I cosmological model filled with viscous fluid, in the presence of a cosmological constant Λ. The role of viscous fluid and Λ term in the evolution the BI spacetime is studied. Though the viscosity cannot remove the cosmological singularity, it plays a crucial part in the formation of a qualitatively new behavior of the solutions near singularity. It is shown that the introduction of the Λ term can be handy in the elimination of the cosmological singularity. In particular, in case of a bulk viscosity, a negative Λ provides a never-ending process of evolution, whereas, for some positive values of Λ and the bulk viscosity being inverse proportional to the expansion, the BI Universe admits a singularity-free oscillatory mode of expansion. In case of a constant bulk viscosity and share viscosity being proportional to expansion, the model allows both non-periodic and inflationary expansion independent to the sign of Λ term.

Keywords: Bianchi type I (BI) model; cosmological constant; viscous fluid.

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1. Introduction

The investigation of relativistic cosmological models usually has the energy–momentum tensor of matter generated by a perfect fluid. To consider more realistic models one must take into account the viscosity mechanisms, which have already attracted the attention of many researchers. Misner[1,2] suggested that strong dissipative due to the neutrino viscosity may considerably reduce the anisotropy of the blackbody radiation. Viscosity mechanism in cosmology can explain the anomalously high entropy per baryon in the present universe.[3,4] Bulk viscosity associated with the grand-unified-theory phase transition[5] may lead to an inflationary scenario.[6–8]

A uniform cosmological model filled with fluid which possesses pressure and second (bulk) viscosity was developed by Murphy.[9] The effect of bulk viscosity on the evolution of a FRW universe at large was investigated in Ref. 10. The solutions that he found exhibit an interesting feature that the big bang type singularity appears in the infinite past. Exact solutions of the isotropic homogeneous cosmology for open, closed and flat universe have been found by Santos *et al.*,[11] with the bulk viscosity being a power function of energy density.

The nature of cosmological solutions for homogeneous Bianchi type I (BI) model was investigated by Belinsky and Khalatnikov[12] by taking into account dissipative process due to viscosity. They showed that viscosity cannot remove the cosmological singularity but results in a qualitatively new behavior of the solutions near singularity. They found the remarkable property that during the time of the *big bang* matter is created by the gravitational field. BI solutions in case of stiff matter with a shear viscosity being the power function of energy density were obtained by Banerjee,[13] whereas BI models with bulk viscosity (η) that is a power function of energy density ε and when the universe is filled with stiff matter were studied by Huang.[14] The effect of bulk viscosity, with a time varying bulk viscous coefficient, on the evolution of isotropic FRW models in the context of open thermodynamics system was studied by Desikan.[15] This study was further developed by Krori and Mukherjee[16] for anisotropic Bianchi models. Cosmological solutions with nonlinear bulk viscosity were obtained in Ref. 17. Models with both shear and bulk viscosity were investigated in Refs. 18 and 19.

Though Murphy[9] claimed that the introduction of bulk viscosity can avoid the initial singularity at finite past, results obtained in Ref. 20 show that it is, in general, not valid, since for some cases big bang singularity occurs in finite past. Using the truncated Israel–Stewart theory of irreversible thermodynamics,[21–23] Coley *et al.*[24–26] described the bulk viscous pressure and anisotropic stress in a class of spatially homogeneous viscous fluid cosmological models. In particular, in case of the BI models they showed that anisotropic stress leads to the models that violate weak energy condition and to the creation of a periodic orbit in some instances. Inflationary BI cosmological models with bulk, shear, and nonlinear viscosity were studied by Grøn.[27] We also mention the recent papers by Pradhan *et al.*[28,29] where the authors studied the Bianchi type I cosmological models with viscous fluid and variable Λ-term and the behavior of magnetic field in BI cosmology for bulk viscous distribution, respectively.

We studied a self-consistent system of the nonlinear spinor and/or scalar fields in a BI spacetime in the presence of a perfect fluid and a Λ term[30,31] in order to clarify whether the presence of a singular point is an inherent property of the relativistic cosmological models or it is only a consequence of specific simplifying assumptions underlying these models. Recently we have considered a nonlinear spinor field in a BI Universe filled with viscous fluid.[32] We also study the role of a magnetic field in the evolution of a BI Universe and interacting spinor and scalar fields.[33] Since the viscous fluid itself presents a growing interest, we study the influence of viscous fluid and Λ term in the evolution of the BI Universe in this report.

2. Derivation of Basic Equations

Using the variational principle in this section we derive the fundamental equations for the gravitational field from the action (2.1):

$$\mathcal{S}(g;\varepsilon) = \int \mathcal{L}\sqrt{-g}\,d\Omega \tag{2.1}$$

with

$$\mathcal{L} = \mathcal{L}_{\text{grav.}} + \mathcal{L}_{\text{vf}}\,. \tag{2.2}$$

The gravitational part of the Lagrangian (2.2) $\mathcal{L}_{\text{grav.}}$ is given by a Bianchi type-I metric, whereas the term $\mathcal{L}_{\text{vf}}$ describes a viscous fluid.

We also write the expressions for the metric functions explicitly in terms of the volume scale τ defined below (2.23). Defining Hubble constant (2.37) in analogy with a flat Friedmann–Robertson–Walker (FRW) Universe, we also derive the system of equations for τ, H and ε, with ε being the energy density of the viscous fluid, which plays the central role here.

2.1. *The gravitational field*

As a gravitational field we consider the Bianchi type I (BI) cosmological model. It is the simplest model of anisotropic universe that describes a homogeneous and spatially flat spacetime and is filled with perfect fluid with the equation of state $p = \zeta\varepsilon$, $\zeta < 1$, it eventually evolves into a FRW universe.[34] The isotropy of present-day universe makes BI model a prime candidate for studying the possible effects of an anisotropy in the early universe on modern-day data observations. In view of what has been mentioned above we choose the gravitational part of the Lagrangian (2.2) in the form

$$\mathcal{L}_{\text{g}} = \frac{R}{2\kappa}\,, \tag{2.3}$$

where R is the scalar curvature, $\kappa = 8\pi G$ being the Einstein's gravitational constant. The gravitational field in our case is given by a Bianchi type I (BI) metric

$$ds^2 = dt^2 - a^2\,dx^2 - b^2\,dy^2 - c^2\,dz^2\,, \tag{2.4}$$

with a, b, c being the functions of time t only. Here the speed of light is taken to be unity.

The metric (2.4) has the following non-trivial Christoffel symbols

$$\begin{aligned}
&\Gamma^1_{10} = \frac{\dot a}{a}\,, \qquad \Gamma^2_{20} = \frac{\dot b}{b}\,, \qquad \Gamma^3_{30} = \frac{\dot c}{c}\,,\\
&\Gamma^0_{11} = a\dot a\,, \qquad \Gamma^0_{22} = b\dot b\,, \qquad \Gamma^0_{33} = c\dot c\,.
\end{aligned} \tag{2.5}$$

The nontrivial components of the Ricci tensors are

$$R^0_0 = -\left(\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c}\right), \tag{2.6a}$$

$$R_1^1 = -\left[\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right)\right], \tag{2.6b}$$

$$R_2^2 = -\left[\frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\left(\frac{\dot{c}}{c} + \frac{\dot{a}}{a}\right)\right], \tag{2.6c}$$

$$R_3^3 = -\left[\frac{\ddot{c}}{c} + \frac{\dot{c}}{c}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)\right]. \tag{2.6d}$$

From (2.6) one finds the following Ricci scalar for the BI universe

$$R = -2\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{\dot{c}}{c}\frac{\dot{a}}{a}\right). \tag{2.7}$$

The nontrivial components of Riemann tensors in this case read

$$\begin{aligned} &R^{01}{}_{01} = -\frac{\ddot{a}}{a}\,, \qquad R^{02}{}_{02} = -\frac{\ddot{b}}{b}\,, \qquad R^{03}{}_{03} = -\frac{\ddot{c}}{c}\,, \\ &R^{12}{}_{12} = -\frac{\dot{a}}{a}\frac{\dot{b}}{b}\,, \qquad R^{23}{}_{23} = -\frac{\dot{b}}{b}\frac{\dot{c}}{c}\,, \qquad R^{31}{}_{31} = -\frac{\dot{c}}{c}\frac{\dot{a}}{a}\,. \end{aligned} \tag{2.8}$$

Now having all the nontrivial components of Ricci and Riemann tensors, one can easily write the invariants of gravitational field which we need to study the space-time singularity. We return to this study at the end of this section.

2.2. *Viscous fluid*

The influence of the viscous fluid in the evolution of the Universe is performed by means of its energy–momentum tensor, which acts as the source of the corresponding gravitational field. The reason for writing $\mathcal{L}_{\rm vf}$ in (2.2) is to underline that we are dealing with a self-consistent system. The energy–momentum tensor of a viscous field has the form

$$T_{\mu\,(\mathrm{m})}^{\nu} = (\varepsilon + p')u_\mu u^\nu - p'\delta_\mu^\nu + \eta g^{\nu\beta}[u_{\mu;\beta} + u_{\beta:\mu} - u_\mu u^\alpha u_{\beta;\alpha} - u_\beta u^\alpha u_{\mu;\alpha}]\,, \tag{2.9}$$

where

$$p' = p - \left(\xi - \frac{2}{3}\eta\right)u^\mu_{;\mu}\,. \tag{2.10}$$

Here ε is the energy density, p pressure, η and ξ are the coefficients of shear and bulk viscosity, respectively. Note that the bulk and shear viscosities, η and ξ, are both positively definite, i.e.

$$\eta > 0\,, \qquad \xi > 0\,. \tag{2.11}$$

They may be either constant or function of time or energy, such as:

$$\eta = |A|\varepsilon^\alpha\,, \qquad \xi = |B|\varepsilon^\beta\,. \tag{2.12}$$

The pressure p is connected to the energy density by means of an equation of state. In this report we consider the one describing a perfect fluid:

$$p = \zeta\varepsilon\,, \qquad \zeta \in (0, 1]\,. \tag{2.13}$$

Note that here $\zeta \neq 0$, since for dust pressure, hence the temperature is zero, that results in vanishing viscosity.

In a comoving system of reference such that $u^\mu = (1, 0, 0, 0)$ we have

$$T_{0\,(\mathrm{m})}^{0} = \varepsilon\,, \tag{2.14a}$$

$$T_{1\,(\mathrm{m})}^{1} = -p' + 2\eta\frac{\dot a}{a}\,, \tag{2.14b}$$

$$T_{2\,(\mathrm{m})}^{2} = -p' + 2\eta\frac{\dot b}{b}\,, \tag{2.14c}$$

$$T_{3\,(\mathrm{m})}^{3} = -p' + 2\eta\frac{\dot c}{c}\,. \tag{2.14d}$$

Let us introduce the dynamical scalars such as the expansion and the shear scalar as usual

$$\theta = u^\mu_{;\mu}\,, \qquad \sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}\,, \tag{2.15}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\alpha}P^\alpha_\nu + u_{\nu;\alpha}P^\alpha_\mu\right) - \frac{1}{3}\theta P_{\mu\nu}\,. \tag{2.16}$$

Here P is the projection operator obeying

$$P^2 = P\,. \tag{2.17}$$

For the spacetime with signature $(+, -, -, -)$ it has the form

$$P_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu\,, \qquad P^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu\,. \tag{2.18}$$

For the BI metric the dynamical scalar has the form

$$\theta = \frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c} = \frac{\dot\tau}{\tau} \tag{2.19}$$

and

$$2\sigma^2 = \frac{\dot a^2}{a^2} + \frac{\dot b^2}{b^2} + \frac{\dot c^2}{c^2} - \frac{1}{3}\theta^2\,. \tag{2.20}$$

2.3. *Field equations and their solutions*

Variation of (2.1) with respect to metric tensor $g_{\mu\nu}$ gives the Einstein's field equation. In account of the Λ-term we then have

$$G^\nu_\mu = R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R = \kappa T^\nu_\mu - \delta^\nu_\mu\Lambda\,. \tag{2.21}$$

In view of (2.6) and (2.7) for the BI spacetime (2.4) we rewrite Eq. (2.21) as

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} = \kappa T_1^1 - \Lambda\,, \tag{2.22a}$$

$$\frac{\ddot{c}}{c} + \frac{\ddot{a}}{a} + \frac{\dot{c}}{c}\frac{\dot{a}}{a} = \kappa T_2^2 - \Lambda\,, \tag{2.22b}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} = \kappa T_3^3 - \Lambda\,, \tag{2.22c}$$

$$\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{\dot{c}}{c}\frac{\dot{a}}{a} = \kappa T_0^0 - \Lambda\,, \tag{2.22d}$$

where over dot means differentiation with respect to t and T_ν^μ is the energy–momentum tensor of a viscous fluid given above (2.14).

2.3.1. *Expressions for the metric functions*

To write the metric functions explicitly, we define a new time-dependent function $\tau(t)$

$$\tau = abc = \sqrt{-g}\,, \tag{2.23}$$

which is indeed the volume scale of the BI spacetime.

Let us now solve the Einstein equations. In account of (2.14) subtracting (2.22a) from (2.22b), (2.22b) from (2.22c) and (2.22c) from (2.22a) one finds

$$\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right) = \frac{X_1}{\tau}\exp\left[-2\kappa\int\eta\,dt\right], \tag{2.24a}$$

$$\left(\frac{\dot{b}}{b} - \frac{\dot{c}}{c}\right) = \frac{X_2}{\tau}\exp\left[-2\kappa\int\eta\,dt\right], \tag{2.24b}$$

$$\left(\frac{\dot{c}}{c} - \frac{\dot{a}}{a}\right) = \frac{X_3}{\tau}\exp\left[-2\kappa\int\eta\,dt\right]. \tag{2.24c}$$

Equation (2.24) implies that as $\tau \to \infty$ the expansion rate becomes isotropic in all directions. A little manipulation of (2.24) leads to the following relations between the metric functions a, b and c:

$$\frac{a}{b} = D_1 \exp\left(X_1 \int \frac{e^{-2\kappa\int\eta\,dt'}}{\tau}\,dt\right), \tag{2.25a}$$

$$\frac{b}{c} = D_2 \exp\left(X_2 \int \frac{e^{-2\kappa\int\eta\,dt'}}{\tau}\,dt\right), \tag{2.25b}$$

$$\frac{c}{a} = D_3 \exp\left(X_3 \int \frac{e^{-2\kappa\int\eta\,dt'}}{\tau}\,dt\right). \tag{2.25c}$$

Here D_1, D_2, D_3, X_1, X_2, X_3 are integration constants, obeying

$$D_1 D_2 D_3 = 1\,, \qquad X_1 + X_2 + X_3 = 0\,. \tag{2.26}$$

In view of (2.26) from (2.25) we write the metric functions explicitly[30]

$$a(t) = A_1 \tau^{1/3}(t) \exp\left[(B_1/3)\int \frac{e^{-2\kappa\int \eta\, dt'}}{\tau(t'')}\, dt''\right], \tag{2.27a}$$

$$b(t) = A_2 \tau^{1/3}(t) \exp\left[(B_2/3)\int \frac{e^{-2\kappa\int \eta\, dt'}}{\tau(t'')}\, dt''\right], \tag{2.27b}$$

$$c(t) = A_3 \tau^{1/3}(t) \exp\left[(B_3/3)\int \frac{e^{-2\kappa\int \eta\, dt'}}{\tau(t'')}\, dt''\right], \tag{2.27c}$$

where

$$A_1 = \sqrt[3]{(D_1/D_3)}\,, \qquad A_2 = \sqrt[3]{1/(D_1^2 D_3)}\,, \qquad \sqrt[3]{(D_1 D_3^2)}\,,$$

$$B_1 = X_1 - X_3\,, \qquad B_2 = -(2X_1 + X_3)\,, \qquad B_3 = X_1 + 2X_3\,.$$

Thus, the metric functions are found explicitly in terms of τ and viscosity.

The integrals in (2.24), (2.25) and (2.27) are indefinite. As one sees from (2.27a), (2.27b) and (2.27c), for $\tau = t^n$ with $n > 1$ the exponent tends to unity as $t \to \infty$, and the anisotropic model becomes isotropic one.

2.3.2. *Singularity analysis*

Let us now investigate the existence of singularity (singular point) of the gravitational case, which can be done by investigating the invariant characteristics of the spacetime. In general relativity these invariants are composed of the curvature tensor and the metric one. In a 4D Riemann spacetime there are 14 independent invariants [cf. e.g. Ref. 30]. Instead of analyzing all 14 invariants, one can confine this study only in 3, namely the scalar curvature $I_1 = R$, $I_2 = R_{\mu\nu}R^{\mu\nu}$, and the Kretschmann scalar $I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$.[35,36] At any regular spacetime point, these three invariants I_1, I_2, I_3 should be finite. Let us rewrite these invariants in detail.

For the Bianchi I metric one finds the scalar curvature

$$I_1 = R = -2\left(\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c} + \frac{\dot a}{a}\frac{\dot b}{b} + \frac{\dot b}{b}\frac{\dot c}{c} + \frac{\dot c}{c}\frac{\dot a}{a}\right). \tag{2.28}$$

Since the Ricci tensor for the BI metric is diagonal, the invariant $I_2 = R_{\mu\nu}R^{\mu\nu} \equiv R_\mu^\nu R_\nu^\mu$ is a sum of squares of diagonal components of Ricci tensor, i.e.

$$I_2 = [(R_0^0)^2 + (R_1^1)^2 + (R_2^2)^2 + (R_3^3)^2]\,, \tag{2.29}$$

with the components of the Ricci tensor being given by (2.6).

Analogically, for the Kretschmann scalar in this case we have $I_3 = R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu}$, a sum of squared components of all nontrivial $R^{\mu\nu}{}_{\mu\nu}$, which in view of (2.8) can be written as

$$\begin{aligned} I_3 &= 4\left[(R^{01}{}_{01})^2 + (R^{02}{}_{02})^2 + (R^{03}{}_{03})^2 + (R^{12}{}_{12})^2 + (R^{23}{}_{23})^2 + (R^{31}{}_{31})^2\right] \\ &= 4\left[\left(\frac{\ddot a}{a}\right)^2 + \left(\frac{\ddot b}{b}\right)^2 + \left(\frac{\ddot c}{c}\right)^2 + \left(\frac{\dot a}{a}\frac{\dot b}{b}\right)^2 + \left(\frac{\dot b}{b}\frac{\dot c}{c}\right)^2 + \left(\frac{\dot c}{c}\frac{\dot a}{a}\right)^2\right]. \end{aligned} \tag{2.30}$$

Let us now express the foregoing invariants in terms of τ. From Eqs. (2.27) we have

$$a_i = A_i \tau^{1/3} \exp\left((B_i/3)\int \frac{e^{-2\kappa\int \eta\, dt}}{\tau(t)}\, dt\right), \tag{2.31a}$$

$$\frac{\dot a_i}{a_i} = \frac{\dot\tau + B_i e^{-2\kappa\int\eta\, dt}}{3\tau}, \qquad i = 1, 2, 3, \tag{2.31b}$$

$$\frac{\ddot a_i}{a_i} = \frac{3\tau\ddot\tau - 2\dot\tau^2 - \dot\tau B_i e^{-2\kappa\int\eta\, dt} - 6\kappa\eta\tau B_i e^{-2\kappa\int\eta\, dt} + B_i^2 e^{-4\kappa\int\eta\, dt}}{9\tau^2}, \tag{2.31c}$$

i.e. the metric functions a, b, c and their derivatives are in functional dependence with τ. From Eqs. (2.31) one can easily verify that

$$I_1 \propto \frac{1}{\tau^2}, \qquad I_2 \propto \frac{1}{\tau^4}, \qquad I_3 \propto \frac{1}{\tau^4}.$$

Moreover, as it can be seen from Ref. 30, the remaining 11 invariants are constructed from either a greater number of Ricci and/or Riemann tensors or their self-dual. So those invariants will be inversely proportional to τ as well. Thus we see that at any spacetime point, where $\tau = 0$ the invariants I_1, I_2 and I_3 together with all the rest of the 11 become infinity, hence the spacetime becomes singular at this point.

2.4. *Equations for determining τ*

In Sec. 2.3 we wrote the corresponding metric functions in terms of volume scale τ. In what follows, we write the equation for τ and study it in details.

Summation of Einstein equations (2.22a), (2.22b), (2.22c) and three times (2.22d) gives

$$\ddot\tau - \frac{3}{2}\kappa\xi\dot\tau = \frac{3}{2}\kappa(\varepsilon - p)\tau - 3\Lambda\tau. \tag{2.32}$$

For the right-hand side of (2.32) to be a function of τ only, the solution to this equation is well known.[37]

From the contracted Bianchi identities $G^{\nu}_{\mu;\nu} = 0$ one finds

$$T^{\nu}_{\mu;\nu} = T^{\nu}_{\mu,\nu} + \Gamma^{\nu}_{\rho\nu}T^{\rho}_{\mu} - \Gamma^{\rho}_{\mu\nu}T^{\nu}_{\rho} = 0, \tag{2.33}$$

which in our case has the form

$$\frac{1}{\tau}(\tau T^0_0)^{\cdot} - \frac{\dot a}{a}T^1_1 - \frac{\dot b}{b}T^2_2 - \frac{\dot c}{c}T^3_3 = 0. \tag{2.34}$$

After a little manipulation from (2.34) we obtain

$$\dot{\varepsilon} + \frac{\dot{\tau}}{\tau}\omega - \left(\xi + \frac{4}{3}\eta\right)\frac{\dot{\tau}^2}{\tau^2} + 4\eta(\kappa T_0^0 - \Lambda) = 0\,, \tag{2.35}$$

where

$$\omega = \varepsilon + p\,, \tag{2.36}$$

is the thermal function.

Let us now, in analogy with Hubble constant in a FRW Universe, introduce a generalized Hubble constant H:

$$\frac{\dot{\tau}}{\tau} = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = 3H\,. \tag{2.37}$$

Then (2.32) and (2.35) in account of (2.14) can be rewritten as

$$\dot{H} = \frac{\kappa}{2}(3\xi H - \omega) - (3H^2 - \kappa\varepsilon + \Lambda)\,, \tag{2.38a}$$

$$\dot{\varepsilon} = 3H(3\xi H - \omega) + 4\eta(3H^2 - \kappa\varepsilon + \Lambda)\,. \tag{2.38b}$$

Equations (2.38) can be written in terms of dynamical scalar as well.

In account of (2.27) one can also rewrite share scalar (2.20) as

$$2\sigma^2 = \frac{6(X_1^2 + X_1 X_3 + X_3^2)}{9\tau^2} e^{-4\kappa\int \eta\, dt}\,. \tag{2.39}$$

From (2.22d) we have

$$\frac{1}{3}\theta^2 - \sigma^2 = \kappa\varepsilon - \Lambda\,. \tag{2.40}$$

Equations (2.38) can now be written in terms of θ and σ as follows:

$$\dot{\theta} = \frac{3\kappa}{2}(\xi\theta - \omega) - 3\sigma^2\,, \tag{2.41a}$$

$$\dot{\varepsilon} = \theta(\xi\theta - \omega) + 4\eta\sigma^2\,. \tag{2.41b}$$

Note that Eqs. (2.41) coincide with the ones given in Ref. 13.

2.5. *Some special solutions*

In this subsection we simultaneously solve the system of equations for τ, H and ε. It is convenient to rewrite the Eqs. (2.37) and (2.38) as a single system:

$$\dot{\tau} = 3H\tau\,, \tag{2.42a}$$

$$\dot{H} = \frac{\kappa}{2}(3\xi H - \omega) - (3H^2 - \kappa\varepsilon + \Lambda)\,, \tag{2.42b}$$

$$\dot{\varepsilon} = 3H(3\xi H - \omega) + 4\eta(3H^2 - \kappa\varepsilon + \Lambda)\,. \tag{2.42c}$$

In account of (2.36), (2.12) and (2.13), Eqs. (2.42) can now be rewritten as

$$\dot{\tau} = 3H\tau\,, \tag{2.43a}$$

$$\dot{H} = \frac{\kappa}{2}(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon) - (3H^2 - \kappa\varepsilon + \Lambda)\,, \tag{2.43b}$$

$$\dot{\varepsilon} = 3H(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon) + 4A\varepsilon^{\alpha}(3H^2 - \kappa\varepsilon + \Lambda)\,. \tag{2.43c}$$

The system (2.42) have been extensively studied in literature either partially[9,14,13] or in general.[12] In what follows, we consider the system (2.42) for some special choices of the parameters.

2.5.1. *Case with bulk viscosity*

Let us first consider the case when the real fluid possesses the bulk viscosity only. The corresponding system of equations can then be obtained by setting $\eta = 0$ in (2.42) or $A = 0$ in (2.43). In this case Eqs. (2.42a) and (2.42b) remain unaltered, while (2.42c) takes the form

$$\dot{\varepsilon} = 3H(3\xi H - \omega)\,. \tag{2.44}$$

In view of (2.44) the system (2.42) admits the following first integral

$$\tau^2(\kappa\varepsilon - 3H^2 - \Lambda) = C_1\,, \qquad C_1 = \text{const.} \tag{2.45}$$

From the relation (2.45) we can draw a few conclusions. At the initial stage of evolution the volume scale τ and the energy density ε are believed to be close to zero and infinity, respectively. In that sense (2.45) does not contradict with the picture presently thought to be the correct one. It is interesting to see how does the relation (2.45) influence the further evolution of the Universe. It is well known that with the expansion of the Universe, i.e. with the increase of τ, the energy density ε decreases. Suppose at some stage of expansion τ becomes so large that $\tau^{-2} \to 0$ and ε becomes too small to be ignored. Then from (2.45) it follows that

$$3H^2 + \Lambda \to 0\,. \tag{2.46}$$

In case of $\Lambda = 0$, we find $H = 0$, i.e. in the absence of a Λ term, once $\tau \to \infty$, the process of evolution is terminated, i.e. τ remains constant with time t. As one sees from (2.46), for the H to make any sense, the Λ term should be negative. In the presence of a negative Λ term the evolution process of the Universe never comes to a halt. Thus we see that the Universe may be infinitely large only if $\Lambda \leq 0$. Moreover, with $\Lambda < 0$ we have an ever-expanding Universe. Recall that in our case a negative Λ corresponds to a universal repulsive force, which nowadays is seen to be a form of dark energy. In that sense (2.45) is also in good agreement with the models of accelerated expansion. It should be mentioned that the relation (2.45) also impose some restriction on τ and ε in case of a positive Λ which corresponds to an additional gravitational force. Expression (2.46) together with (2.45) show that in case of $\Lambda > 0$ the volume scale τ cannot be infinitely large (or ε cannot be too small to be ignored), i.e. τ is bound from above as well. This corresponds to the earlier results (oscillatory mode of evolution) obtained by us.[30,31]

Let us now consider the case when the bulk viscosity is inverse proportional to expansion, i.e.

$$\xi\theta = C_2\,, \qquad C_2 = \text{const.} \tag{2.47}$$

Now keeping into mind that $\theta = \dot\tau/\tau = 3H$, also the relations (2.42a), (2.36) and (2.13), Eq. (2.44) can be written as

$$\frac{\dot\varepsilon}{C_2 - (1+\zeta)\varepsilon} = \frac{\dot\tau}{\tau}\,. \tag{2.48}$$

From Eq. (2.48) one finds

$$\varepsilon = \frac{1}{1+\zeta}[C_2 + C_3\tau^{-(1+\zeta)}]\,, \tag{2.49}$$

with C_3 being some arbitrary constant. Further, inserting ε from (2.49) into (2.32) one finds the expression for τ explicitly.

Taking into account the equation of state (2.13) in view of (2.47) and (2.49), Eq. (2.32) admits the following solution in quadrature:

$$\int \frac{d\tau}{\sqrt{C_2^2 + C_0^0\tau^2 + C_1^1\tau^{1-\zeta}}} = t + t_0\,, \tag{2.50}$$

where C_2^2 and t_0 are some constants. Further we set $t_0 = 0$. Here, $C_0^0 = 3\kappa C_2/(1+\zeta) - 3\Lambda$ and $C_1^1 = 3\kappa C_3/(1+\zeta)$. As one sees, C_0^0 is negative for

$$\Lambda > \kappa C_2/(1+\zeta)\,. \tag{2.51}$$

It means that for a positive Λ obeying (2.51) (we assume that the constant C_2 is positive) τ should be bound from above as well. Let us now rewrite Eq. (2.50) in the form

$$\dot\tau = \sqrt{2[E - \mathcal{U}(\tau)]}\,, \tag{2.52}$$

where $E = C_2^2/2$ can be viewed as energy and $\mathcal{U}(\tau) = -0.5(C_0^0\tau^2 + C_1^1\tau^{1-\zeta})$ can be seen as the potential (cf. Fig. 1) corresponding to Eq. (2.32). Depending on the value of E there exists two types of solutions: for $E > 0$ we have non-periodic solutions, i.e. after reaching some maximum value (say $\tau_{\max}$) the BI Universe begins to contract and finally collapses into a point, thus giving rise to a spacetime singularity; for $E < 0$ BI spacetime admits a singularity-free oscillatory mode of expansion (cf. Fig. 2). A comprehensive description concerning potential can be found in Ref. 31. Thus we see that, in case of bulk viscosity alone, a positive Λ gives rise to a model that is bound from above.

As a second example we consider the case, when $\zeta = 1$. From (2.50) one then finds

$$\tau(t) = \begin{cases} (\exp(\sqrt{C_0^0}\,t) - C_2^2\exp(\sqrt{C_0^0}\,t))/(2\sqrt{C_0^0})\,, & C_0^0 > 0\,, \quad \text{(2.53a)} \\ (C_2^2/\sqrt{|C_0^0|})\sin(\sqrt{|C_0^0|}\,t)\,, & C_0^0 < 0\,. \quad \text{(2.53b)} \end{cases}$$

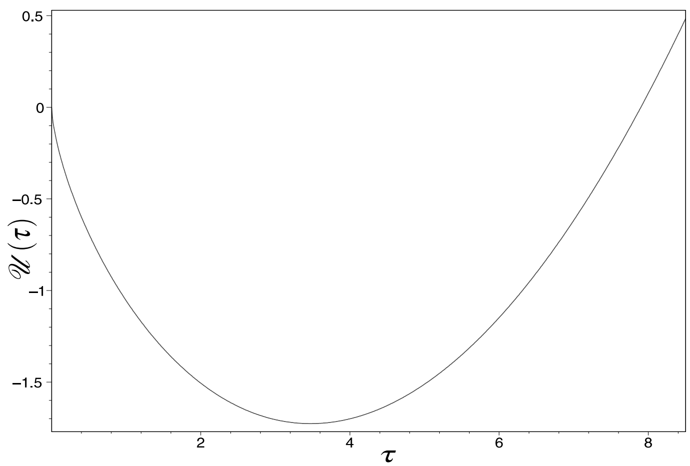

Fig. 1. View of the potential $\mathcal{U}(\tau)$. Here we set $\kappa = 1$, $C_2 = 1$ and $C_3 = 1$. Perfect fluid corresponds to a radiation, i.e. $\zeta = 0.33$, and the cosmological constant is taken to be $\Lambda = 0.8$.

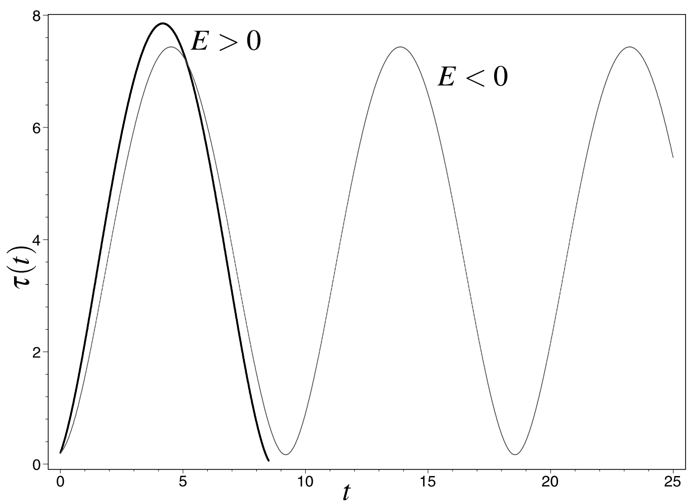

Fig. 2. Evolution of the BI spacetime corresponding to the potential given in Fig. 1. The initial value of the volume scale in this case is taken to be $\tau_0 = 0.2$.

Taking into account that $C_0^0 > 0$ for any non-positive Λ, from (2.53a) one sees that, in case of $\Lambda \leq 0$ the Universe may be infinitely large (there is no upper bound), which is in line with the conclusion made above. On the other hand, C_0^0 may be negative only for some positive value of Λ. It was shown in Refs. 30 and 31 that in case of a perfect fluid a positive Λ always invokes oscillations in the model, whereas, in the present model with viscous fluid, it is the case only when Λ obeys (2.51).

Unlike the case with radiation where BI admits two types of solutions, the case with stiff matter allows only one type of solutions, namely the non-periodic one that corresponds to $E > 0$ in the previous case, since now potential $\mathcal{U}(\tau) = -0.5C_0^0\tau^2$ has its minimum at $\tau = 0$.

2.5.2. *Case with shear and bulk viscosity*

Let us now consider the general case with the shear viscosity η being proportional to the expansion, i.e.

$$\eta \propto \theta = 3H\,. \tag{2.54}$$

We will consider the case when

$$\eta = -\frac{3}{2\kappa}H\,. \tag{2.55}$$

In this case from (2.42b) and (2.42c) one easily find

$$3H^2 = \kappa\varepsilon + C_4\,, \qquad C_4 = \text{const.} \tag{2.56}$$

From (2.56) it follows that at the initial state of expansion, when ε is large, the Hubble constant is also large and with the expansion of the Universe H decreases as does ε. As H^2 and ε are both non-negative, we may conclude that the constant C_4 is non-negative as well. Inserting the relation (2.56) into the Eqs. (2.42b) one finds

$$\int \frac{dH}{AH^2 + BH + C} = t\,, \tag{2.57}$$

where, $A = -1.5(1+\zeta)$, $B = 1.5\kappa\xi$ and $C = 0.5C_4(\zeta - 1) - \Lambda$. If the bulk viscosity is taken to be a constant, i.e. $\xi = \text{const.}$, then depending on the value of the discriminant $B^2 - 4AC$ there exist three types of solutions, namely[38]:

$$t = \begin{cases} \dfrac{1}{\sqrt{B^2-4AC}} \ln\left|\dfrac{2AH + B + \sqrt{B^2-4AC}}{2AH + B - \sqrt{B^2-4AC}}\right|, & B^2 > 4AC\,, \\[2ex] \dfrac{2}{\sqrt{4AC - B^2}} \arctan \dfrac{2AH + B}{\sqrt{4AC - B^2}}\,, & B^2 < 4AC\,, \\[2ex] -\dfrac{2}{2AH + B}\,, & B^2 = 4AC\,. \end{cases} \tag{2.58}$$

In view of the fact that $A < 0$ and $C_4(\zeta - 1) \leq 0$ the sign of C depends on the sign and value of Λ. In this case the system allows non-periodic or exponential expansion of τ for both positive and negative Λ. As one can see from Fig. 3, independent to the sign of Λ model allows solutions which initially expand, reach the maximum and then begin to contract finally giving rise to a spacetime singularity. On the other hand, with the same Λ's but different κ and ξ the expansion of BI Universe takes exponential form (cf. Fig. 4).

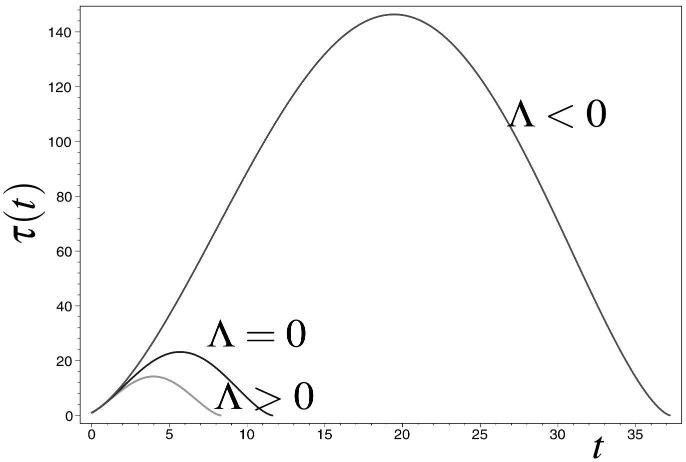

Fig. 3. View of $\tau(t)$ for $C_4 = 0.1$, $\zeta = 0.33$, $\kappa = 0.1$ and $\xi = 0.1$ with $\Lambda = -0.03$, $\Lambda = 0$, and $\Lambda = 0.03$, respectively.

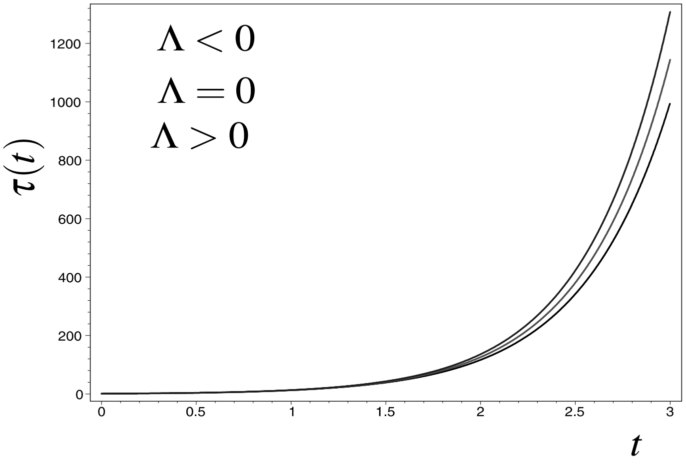

Fig. 4. Evolution of the BI spacetime with the parameters $C_4 = 0.1$, $\zeta = 0.33$, $\kappa = 1$ and $\xi = 1$ with $\Lambda = -0.03$, $\Lambda = 0$, and $\Lambda = 0.03$, as in previous case.

It should be noted that in case of a perfect fluid models with positive Λ give either oscillatory or non-periodic solutions, while for a negative Λ expansion is always exponential. Inclusion of viscous fluid gives rise to both non-periodic and inflationary expansion independent to the sign of Λ term. In fact the value of κ and ξ are crucial in the formation of this or that type of solutions.

Finally we give the phase diagrams corresponding to the solutions which are bound from above (cf. Figs. 5 and 6). It should be noted that depending on the problem parameters, the solutions may be sinusoidal as well. A slightly deformed

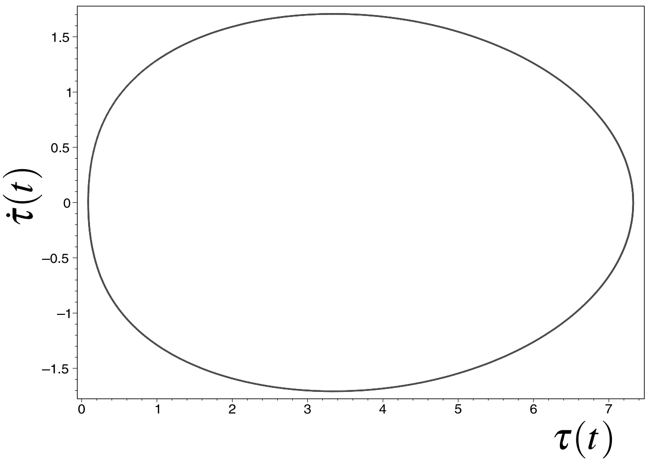

Fig. 5. Phase diagram of the solution given in Fig. 2.

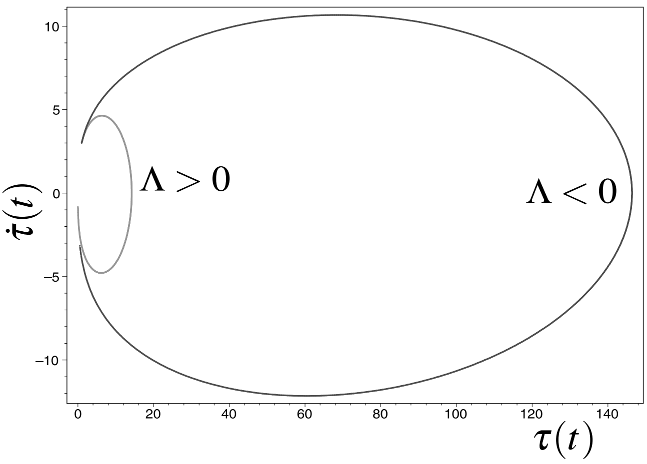

Fig. 6. Phase diagram of the solutions presented in Fig. 3.

shape of the phase diagrams illustrates the non-sinusoidal character of the solutions for the given set of parameters.

3. Conclusion

The role of viscous fluid and Λ term in the evolution of a homogeneous, anisotropic Universe given by a Bianchi type-I spacetime is studied. It is shown that the Λ term plays very important role in BI cosmology. In particular, in case of a bulk viscosity, it provides an everlasting process of evolution with Λ being negative. If the bulk viscosity is inverse proportional to expansion, a positive Λ generates cosmological models which are bounded from above. Moreover, depending on the choice of integration constants, which in this particular case can be viewed as energy level, it may be either closed (corresponds to the non-periodic solution with a spacetime singularity), or an open one (admits to the oscillatory mode of expansion). It should be noted that in case of a perfect fluid only a positive Λ is responsible for non-periodic solutions, whereas, with a viscous fluid the solutions in questions arise even without a Λ term. In particular, if the spacetime is filled with a viscous fluid where the share viscosity is proportional to the expansion and the bulk is a constant, we find both non-periodic and exponential mode of expansion that completely depends on the value of κ and ξ. The Λ-term in this case plays secondary role: a negative Λ gives rise to a greater amplitude for the non-periodic mode, whereas, in case of exponential mode it provides a greater rate of expansion. Here we only consider some special cases which provide exact solutions. For a better knowledge about the evolution, it is important to perform some qualitative analysis of the system (2.42). A detailed analysis of the system in question plus some numerical solutions will be presented soon elsewhere.

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