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Bianchi type-I cosmological model and Bel-Robinson energy tensors

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Abstract

Given the growing interest of the researchers in cosmological models and their eagerness to construct the local tensors describing the strength of gravitational field, we write the expression for the Bel-Robinson tensor defined in three different ways for a Bianchi type-I cosmological model. In doing so we use the orthonormal basis and compare the results obtained with those for ordinary coordinate basis.

Keywords: Bianchi type-I model, Bel-Robinson tensors
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1 Introduction

The spatially homogeneous isotropic Friedmann-Robertson-Walker (FRW) models are widely considered as a good approximation of the present and early stages of the Universe. However, the large scale matter distribution in the observable Universe, largely manifested in the form of discrete structures, does not exhibit a high degree of homogeneity. Recent space investigations detect anisotropy in the cosmic microwave background. The Cosmic Background Explorer's differential radiometer has detected and measured cosmic microwave background anisotropies at different angular scales.

These anisotropies are supposed to contain in their fold the entire history of cosmic evolution dating back to the recombination era and are being considered as indicative of the geometry and the content of the Universe. More information about cosmic microwave background anisotropy is expected to be uncovered by the investigations of the microwave anisotropy probe. There is widespread consensus among cosmologists that cosmic microwave background anisotropies at small angular scales are the key to the formation of discrete structures. The theoretical arguments [1] and recent experimental data that support the existence of an anisotropic phase that approaches an isotropic phase leads one to consider universe models with an anisotropic background. Its simplicity, the Kasner-universe-like behavior near the singularity [2] and evolution into a FRW universe when filled with matter obeying the equation of state $p = \zeta\,\varepsilon, \quad \zeta < 1$ [3] make the Bianchi type-I (BI) model a prime candidate for studying the possible effects of an anisotropy in the early Universe on present-day observations.

On the other hand, the lack of a well-posed definition of *local energy-momentum tensor* which is the consequence of the Principle of Equivalence [4], lies at the heart of Einstein's theory of general relativity and quest for the local tensors describing the strength of gravitational field has long been going on. One of the first successful attempt to address this problem was taken by Bel [5]-[7] and independently Robinson [8]. In recent years tensors describing the strength of gravitational field known as super-energy tensors or Bel-Robinson (BR) tensors have been constructed and their properties were studied by many authors, e. g. [9]-[19]. In this report we give a review of different definitions of BR commonly found in literature and illustrate the components of BR for a BI spacetime.

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2 Bianchi I Universe: a brief description

A diagonal BI spacetime is a spatially homogeneous spacetime, which admits an Abelian group G_3, acting on spacelike hypersurfaces, generated by the spacelike Killing vectors $\xi_i = \partial_i,\ i = 1, 2, 3$ with the Lie algebra $[\xi_i, \xi_j] = 0,\ i, j = 1, 2, 3$. In synchronous coordinates, the metric is [20, 21]:

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i^2(t) dx_i^2. \tag{1}$$

If the three scale factors are equal (i.e., $a_1 = a_2 = a_3$), Eq. (1) describes an isotropic and spatially flat FRW universe. The BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. Thus, a BI universe, being the straightforward generalization of the flat FRW universe, is one of the simplest models of an anisotropic universe that describes a homogeneous and spatially flat universe. When two of the metric functions are equal (e.g., $a_2 = a_3$) the BI spacetime is reduced to the important class of plane symmetric spacetime (a special class of the locally rotational symmetric spacetimes [22, 23]), which admits a G_4 group of isometries acting multiply transitively on the spacelike hypersurfaces of homogeneity generated by the Killing vectors $\xi_i,\ i = 1, 2, 3$, and $\xi_4 = x_2\partial_3 - x_3\partial_2$. The BI has the agreeable property that near the singularity it behaves like a Kasner universe, given by

$$a_1(t) = a_1^0 t^{p_1}, \quad a_2(t) = a_2^0 t^{p_2}, \quad a_3(t) = a_3^0 t^{p_3}, \tag{2}$$

with p_j being the parameters of the BI spacetime which measure the relative anisotropy between any two asymmetry axes. The Kasner solution is a general spatially homogeneous vacuum solution of Einstein's equations, aside from Minkowski metric, where the parameters p_j satisfy the constraints

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1. \tag{3}$$

As one sees, p_1, p_2 and p_3 cannot be equal. Only two of them can be equal, and only in two special cases, namely, $(0, 0, 1)$ and $(-1/3, 2/3, 2/3)$. In all other cases p_1, p_2 and p_3 are different, moreover, one of them is negative, while the two others are positive. If it is supposed that $p_1 < p_2 < p_3$, then their values are confined in the following intervals:

$$-1/3 \leq p_1 \leq 0, \quad 0 \leq p_2 \leq 2/3, \quad 2/3 \leq p_3 \leq 1.$$

The solutions of the algebraic equations (3) can be presented as

$$p_1 = \frac{-p}{p^2 + p + 1}, \quad p_2 = \frac{p(p+1)}{p^2 + p + 1}, \quad p_3 = \frac{p+1}{p^2 + p + 1}.$$

with $0 \leq p \leq 1$. Therefore the space of solutions (modulo isometries) is characterized by only one arbitrary parameter taking values on a circle in $(p_1, p_2, p_3) \in \mathbb{R}^3$.

For later convenience we list the Christoffel symbol, scalar curvature, Ricci, Riemann and Weyl tensors for the BI spacetime. The non-trivial Christoffel symbols are the following:

$$\Gamma^i_{i0} = \frac{\dot{a}_i}{a_i}, \quad \Gamma^0_{ii} = a_i \dot{a}_i, \quad i = 1, 2, 3\,, \tag{4}$$

and the non-trivial components of Riemann tensors are

$$R^{0i}{}_{0i} = -\frac{\ddot{a}_i}{a_i}, \quad R^{ij}{}_{ij} = -\frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j}, \quad i, j = 1, 2, 3, \quad i \neq j. \tag{5}$$

Finally the nontrivial components of the Ricci tensors for the BI metric are

$$R_0^0 = -\sum_{i=1}^{3} \frac{\ddot{a}_i}{a_i}, \quad R_i^i = -\Big[\frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_i}{a_i}\Big(\frac{\dot{a}_j}{a_j} + \frac{\ddot{a}_k}{a_k}\Big)\Big], \qquad i, j, k = 1, 2, 3, \quad i \neq j \neq k.$$

and the scalar curvature is

$$R = -2\Big(\sum_{i=1}^{3}\frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}\Big). \tag{6}$$

It is convenient to separate the Riemann tensor into a trace-free part and a "Ricci" part. This gives the Weyl tensor

$$\begin{aligned} C_{ijkl} &= R_{ijkl} - \frac{1}{(n-2)}\Big(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}\Big) \\ &+ \frac{1}{(n-1)(n-2)}\Big(g_{ik}g_{jl} - g_{il}g_{jk}\Big)R. \end{aligned}$$

This tensor has manifestly all the symmetries of the Riemann tensor. However contrary to the Riemann tensor while it gives rise to Ricci tensor, the Weyl tensor satisfies

$$g^{ik}C_{ijkl} \equiv 0. \tag{7}$$

A further distinction is that while the Riemann tensor can be defined in a manifold endowed only with a connection, the Weyl tensor can be defined only when a metric is also defined. In 4 dimensions the Riemann tensor has 20 distinct components, while the Weyl and the Ricci have 10 components each. The non-trivial components of the Weyl tensor for the BI spacetime are

$$\begin{aligned} C_{0i0i} &= \frac{a_i}{6a_j a_k}\{2\ddot{a}_i a_j a_k - \ddot{a}_j a_k a_i - \ddot{a}_k a_i a_j - \dot{a}_i\dot{a}_j a_k - \dot{a}_k\dot{a}_i a_j + 2\dot{a}_j\dot{a}_k a_i\} \\ C_{jkjk} &= -\frac{a_j a_k}{6a_i}\{2\ddot{a}_i a_j a_k - \ddot{a}_j a_k a_i - \ddot{a}_k a_i a_j - \dot{a}_i\dot{a}_j a_k - \dot{a}_k\dot{a}_i a_j + 2\dot{a}_j\dot{a}_k a_i\}, \\ & i,j,k = 1,2,3, \quad i \neq j \neq k. \end{aligned} \tag{8}$$

From (8) one easily finds the following relation:

$$C_{0i0i} = -\frac{a_i^2}{a_j^2 a_k^2}C_{jkjk}, \quad i,j,k = 1,2,3, \quad i \neq j \neq k. \tag{9}$$

Now having all the non-trivial components of Ricci and Riemann tensors, one can easily write the invariants of gravitational field which we need to study the spacetime singularity. Moreover now we can construct the BR tensor that is defined differently by different authors.

3 Bel-Robinson tensors: definitions and general properties

BR tensor first appeared in the endless search for a covariant version of gravitational energy. As it was mentioned earlier, search for a well-posed definition of local energy-momentum tensor in gravity led Bel [5]-[7] and independently Robinson [8] to construct a four-index tensor for the gravitational field in vacuum.

In general relativity, the energetic content of an electromagnetic field propagating in a region free of charge is described by the well-known symmetric trace-less tensor

$$T_{\rm el}^{\alpha\beta} = -\frac{1}{4\pi}(F^{\alpha\lambda}F_{\lambda}^{\beta} - \frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}), \tag{10}$$

where $F^{\alpha\beta}$ is the electromagnetic field tensor. This tensor satisfies:

$$T_{\rm el\,;\alpha}^{\alpha\beta} = 0 \tag{11}$$

as a consequence of Maxwell equations with $j^{\mu} = 0$. The tensor $T_{\rm el}^{\alpha\beta}$ enables us to define a local density of electromagnetic energy as measured by an observer moving with the unit 4-velocity u:

$$w_{\rm el}(u) = T_{\rm el}^{\alpha\beta}u_{\alpha}u_{\beta}. \tag{12}$$

It follows from (10) that the energy density is positive definite for any time-like vector u.

Within the scope of general relativity, however, it is well known that the concept of local energy density is meaningless for a gravitational field. To overcome this difficulty led to introduce the notion of super-energy tensor constructed with the curvature tensor $R_{\mu\nu\alpha\beta}$. The first example of such a tensor was exhibited by Bel [5], that was further generalized to the case of an arbitrary gravitational field [6]. Note that a similar tensor was also introduced by Robinson [8]. This tensor is now commonly know as the BR tensor as well. Since we are going to compare some distinct definition of BR in this paper, before defining them let us see what kind of properties they should have. The properties of the BR tensor are similar to the traditional energy-momentum tensor and following Senovilla [9, 10] can be formulated as follows: (i) it possesses a positive-definite time-like component and a "causal" momentum vector; (ii) its divergence vanishes (in vacuum); (iii) the tensor is zero if and only if the curvature of the spacetime vanishes; (iv) it has positivity property similar to the electromagnetic one; and some others. The symmetry properties can be written as follows:

$$B_{\mu\nu\alpha\beta} = B_{\nu\mu\alpha\beta}, \tag{13}$$

$$B_{\mu\nu\alpha\beta} = B_{\mu\nu\beta\alpha}, \tag{14}$$

$$B_{\mu\nu\alpha\beta} = B_{\alpha\beta\mu\nu}. \tag{15}$$

These symmetry properties leads to the fact that that in n-dimensional case there are $n(n+1)[n(n+1)+2]/8$ independent components of the BR tensor. In case of $n=4$ out of 256 components only 55 are linearly independent.

In literature there are a few definitions of BR. Here we mention only three.

I. By analogy with the tensor (10) which may be written as

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} + *F_{\mu\alpha} * F_{\nu}^{\alpha}, \tag{16}$$

the BR tensor is defined as [11]:

$$B_{\mu\nu\alpha\beta} = R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}R_{\rho\nu\sigma\beta} + *R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} * R_{\rho\nu\sigma\beta}. \tag{17}$$

Here the dual curvature is $*R^{\mu\nu}{}_{\lambda\sigma} \equiv (1/2)\,\epsilon^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\sigma}$. It should be noted that this definition is adequate only in 4 dimensions and in vacuum. Otherwise this tensor cannot satisfy the so called dominant energy property [24] and therefore this expression should not be used in other dimensions or in non-Ricci-flat spacetimes.

Using the definition of dual curvature, from (17) we find

$$B_{\mu\nu\alpha\beta} = R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}R_{\rho\nu\sigma\beta} + R^{\rho}{}_{\mu}{}^{\sigma}{}_{\beta}R_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}R_{\alpha}{}^{\rho\sigma\tau}R_{\beta\rho\sigma\tau}. \tag{18}$$

The properties (13) and (14) follow immediately from (17) thanks to the symmetry property of Riemann tensor. The property (15) is straightforward from (17), but for (18) it requires

$$g_{\mu\nu}R_{\alpha}{}^{\rho\sigma\tau}R_{\beta\rho\sigma\tau} = g_{\alpha\beta}R_{\mu}{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}. \tag{19}$$

II. The restriction that arises above is due to the fact that in defining the BR tensor we used the dual term with the duality operator acting only on the left pair of indices. To avoid this restrictions the BR tensor can be defined by [13, 25]

$$\begin{aligned} 2B_{\mu\nu\alpha\beta} &= R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}R_{\rho\nu\sigma\beta} + *R^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} * R_{\rho\nu\sigma\beta} \\ &+ R*^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha}\, R*_{\rho\nu\sigma\beta} + *\,R*^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} *R*_{\rho\nu\sigma\beta}, \end{aligned} \tag{20}$$

where the duality operator acts on the left or on the right pair of indices according to its position. Nowadays this is known as the Bel tensor and was introduced by Bel [6] in a slightly different form.

III. Here we give another definition that gives rise to BR tensor, that is trace-less and totally symmetric. It can be achieved by constructing BR by means of Weyl tensor [14, 26].

$$B_{\mu\nu\alpha\beta} = C^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} C_{\rho\nu\sigma\beta} + *C^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} * C_{\rho\nu\sigma\beta}. \tag{21}$$

It can be shown that this BR is totally symmetric, i.e.,

$$B_{ijkl} = B_{(ijkl)}, \tag{22}$$

Moreover, the BR defined through Weyl tensor is trace-free, i.e.,

$$g^{jl} B_{ijkl} \equiv 0. \tag{23}$$

Let us study this case in detail. Using the properties of Levi-Civita tensor we first rewrite 21 in the form

$$B_{\mu\nu\alpha\beta} = C^{\rho}{}_{\mu}{}^{\sigma}{}_{\alpha} C_{\rho\nu\sigma\beta} + C^{\rho}{}_{\mu}{}^{\sigma}{}_{\beta} C_{\rho\nu\sigma\alpha} - \frac{1}{2} g_{\mu\nu} C_{\alpha}{}^{\rho\sigma\tau} C_{\beta\rho\sigma\tau}. \tag{24}$$

In what follows we write the expressions for the components of the BR tensor for a BI metric.

4 BR in BI cosmology

Let us now write the components of BR for a BI metric. In doing so we first write the nontrivial components of Riemann and Weyl tensor corresponding to the BI metric (1) in an orthonormal basis.

The nontrivial components of the Riemann tensor in the orthonormal basis take the form:

$$R_{0i0i} = \frac{\ddot{a}_i}{a_i}, \quad R_{ijij} = -\frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j}, \quad i \neq j = 1, 2, 3.$$

For the nontrivial components of the Weyl tensor in the orthonormal basis we find:

$$\begin{aligned} C_{jkjk} &= -C_{0i0i} = -\frac{2\ddot{a}_i a_j a_k - \ddot{a}_j a_k a_i - \ddot{a}_k a_i a_j - \dot{a}_i \dot{a}_j a_k - \dot{a}_k \dot{a}_i a_j + 2\dot{a}_j \dot{a}_k}{6 a_i a_j a_k}, \\ & \quad i \neq j \neq k = 1, 2, 3. \end{aligned}$$

Once we have the components of Riemann and Weyl tensor, we can now write the components for the BR. In what follows we write the nontrivial components of the BR tensor. In doing so we use all the three definitions mentioned above. Here we note that the sub-scripts i, j, k run from 1 to 3 and they are different, i.e., $i, j, k = 1, 2, 3$, and $i \neq j \neq k$.

I. From (18) we now write

$$B_{iijj} = \frac{\dot{a}_j^2}{a_j^2}\Big[\frac{\dot{a}_i^2}{a_i^2} - \frac{\dot{a}_k^2}{a_k^2}\Big] - \frac{\ddot{a}_j^2}{a_j^2}, \tag{25}$$

$$B_{jjjj} = \frac{\dot{a}_j^2}{a_j^2}\Big[\frac{\dot{a}_i^2}{a_i^2} + \frac{\dot{a}_k^2}{a_k^2}\Big] + \frac{\ddot{a}_j^2}{a_j^2}, \tag{26}$$

$$B_{ijij} = \frac{\dot{a}_i}{a_i}\frac{\dot{a}_j}{a_j}\frac{\dot{a}_k^2}{a_k^2} + \frac{\ddot{a}_i}{a_i}\frac{\ddot{a}_j}{a_j}, \tag{27}$$

$$B_{00jj} = \frac{\dot{a}_j^2}{a_j^2}\Big[\frac{\dot{a}_i^2}{a_i^2} + \frac{\dot{a}_k^2}{a_k^2}\Big] - \frac{\ddot{a}_j^2}{a_j^2}, \tag{28}$$

$$B_{jj00} = -\frac{\ddot{a}_j^2}{a_j^2} + \frac{\ddot{a}_k^2}{a_k^2} + \frac{\ddot{a}_i^2}{a_i^2}, \tag{29}$$

$$B_{j0j0} = -\frac{\dot{a}_j}{a_j}\Big[\frac{\dot{a}_k}{a_k}\frac{\ddot{a}_k}{a_k} + \frac{\dot{a}_i}{a_i}\frac{\ddot{a}_i}{a_i}\Big], \tag{30}$$

$$B_{0000} = \sum_{j=1}^{3} \frac{\ddot{a}_j^2}{a_j^2}. \tag{31}$$

Comparing these expressions with those of [19] one sees that the use of the orthonormal basis in this case does not at all simplify them. As in that case, here too we get the following restrictions on metric functions, that reads

$$\Bigl(\frac{\ddot a_i}{a_i}\Bigr)^2 \pm \Bigl(\frac{\ddot a_j}{a_j}\Bigr)^2 = \Bigl(\frac{\dot a_k}{a_k}\Bigr)^2 \Bigl[\Bigl(\frac{\dot a_j}{a_j}\Bigr)^2 \pm \Bigl(\frac{\ddot a_i}{a_i}\Bigr)^2\Bigr]. \tag{32}$$

It was shown in [19] that that if one defines BR tensor as (17) or (18), it correspond to the Einstein equations with the source field given by a vacuum.

II. Let us write the nontrivial components of BR defined as (20).

$$\begin{aligned}
B_{0000} &= B_{iiii} = \frac{1}{2}\Bigl[\frac{\ddot a_i^2}{a_i^2} + \frac{\ddot a_j^2}{a_j^2} + \frac{\ddot a_k^2}{a_k^2} + \frac{\dot a_i^2}{a_i^2}\frac{\dot a_j^2}{a_j^2} + \frac{\dot a_j^2}{a_j^2}\frac{\dot a_k^2}{a_k^2} + \frac{\dot a_k^2}{a_k^2}\frac{\dot a_i^2}{a_i^2}\Bigr], && (33)\\
B_{00kk} &= -B_{iijj} = \frac{1}{2}\Bigl[\frac{\ddot a_i^2}{a_i^2} + \frac{\ddot a_j^2}{a_j^2} - \frac{\ddot a_k^2}{a_k^2} - \frac{\dot a_i^2}{a_i^2}\frac{\dot a_j^2}{a_j^2} + \frac{\dot a_j^2}{a_j^2}\frac{\dot a_k^2}{a_k^2} + \frac{\dot a_k^2}{a_k^2}\frac{\dot a_i^2}{a_i^2}\Bigr], && (34)\\
B_{ijij} &= \frac{\ddot a_i}{a_i}\frac{\ddot a_j}{a_j} + \frac{\dot a_i}{a_i}\frac{\dot a_j}{a_j}\frac{\dot a_k^2}{a_k^2}, && (35)\\
B_{0i0i} &= -\frac{\dot a_i}{a_i}\Bigl[\frac{\dot a_j}{a_j}\frac{\ddot a_j}{a_j} + \frac{\dot a_k}{a_k}\frac{\ddot a_k}{a_k}\Bigr]. && (36)
\end{aligned}$$

Comparing those expression with those given in [19] we find that the new basis simplifies our task. In this case we have $B_{00kk} = -B_{iijj}$ which was not the case in an ordinary basis.

III. Let us now write the components of BR defined in (21).

$$\begin{aligned}
B_{0000} &= B_{iiii} = \frac{1}{6}\Bigl[\frac{\ddot a_i^2}{a_i^2} + \frac{\ddot a_j^2}{a_j^2} + \frac{\ddot a_k^2}{a_k^2} + \frac{\dot a_i}{a_i}\frac{\dot a_j}{a_j}\frac{\dot a_k}{a_k}\Bigl(\frac{\dot a_i}{a_i} + \frac{\dot a_j}{a_j} + \frac{\dot a_k}{a_k}\Bigr)\\
&- \frac{\dot a_i}{a_i}\frac{\dot a_j}{a_j}\Bigl(\frac{\ddot a_i}{a_i} + \frac{\ddot a_j}{a_j}\Bigr) - \frac{\dot a_j}{a_j}\frac{\dot a_k}{a_k}\Bigl(\frac{\ddot a_j}{a_j} + \frac{\ddot a_k}{a_k}\Bigr) - \frac{\dot a_k}{a_k}\frac{\dot a_i}{a_i}\Bigl(\frac{\ddot a_k}{a_k} + \frac{\ddot a_i}{a_i}\Bigr)\\
&+ \frac{\dot a_i^2}{a_i^2}\frac{\dot a_j^2}{a_j^2} + \frac{\dot a_j^2}{a_j^2}\frac{\dot a_k^2}{a_k^2} + \frac{\dot a_k^2}{a_k^2}\frac{\dot a_i^2}{a_i^2} - \frac{\ddot a_i}{a_i}\frac{\ddot a_j}{a_j} - \frac{\ddot a_j}{a_j}\frac{\ddot a_k}{a_k} - \frac{\ddot a_k}{a_k}\frac{\ddot a_i}{a_i}\\
&+ 2\Bigl(\frac{\ddot a_i}{a_i}\frac{\dot a_j}{a_j}\frac{\dot a_k}{a_k} + \frac{\ddot a_j}{a_j}\frac{\dot a_k}{a_k}\frac{\dot a_i}{a_i} + \frac{\ddot a_k}{a_k}\frac{\dot a_i}{a_i}\frac{\dot a_j}{a_j}\Bigr)\Bigr], && (37)\\
B_{00kk} &= -B_{iijj} = -\frac{1}{18}\Bigl(\frac{\dot a_i}{a_i}\frac{\dot a_j}{a_j} - 2\frac{\dot a_j}{a_j}\frac{\dot a_k}{a_k} + \frac{\dot a_k}{a_k}\frac{\dot a_i}{a_i} - 2\frac{\ddot a_i}{a_i} + \frac{\ddot a_j}{a_j} + \frac{\ddot a_k}{a_k}\Bigr)\\
&\times \Bigl(\frac{\dot a_i}{a_i}\frac{\dot a_j}{a_j} + \frac{\dot a_j}{a_j}\frac{\dot a_k}{a_k} - 2\frac{\dot a_k}{a_k}\frac{\dot a_i}{a_i} + \frac{\ddot a_i}{a_i} - 2\frac{\ddot a_j}{a_j} + \frac{\ddot a_k}{a_k}\Bigr) = -B_{ijij}. && (38)
\end{aligned}$$

In this case though the expressions for the components of BR does not undergo any radical changes, as it was shown in [19], the dominant energy property of the BR now fulfills.

5 Conclusions and further questions

In view of the importance of the BI model in the study of the present day Universe we considered the most simple model with a perfect fluid as a source field. We have investigated the Bel-Robinson tensor for the BI spacetime using the orthonormal basis. Comparing the expressions with those found in [19] we see at least for two cases the new basis significantly simplifies the expression and gives clear idea about the dominant property of BR.

The averaging problem in general relativity is an important issue with many implications in cosmology and in the understanding of the recent expansion history of the visible Universe. We must remind that the averaging methods are far from unique and the problem of defining a suitable averaging scheme remains open [28]. The effects of spatial anisotropies on cosmologies by looking at the average properties of BI models deserve further studies.

As it is known, in curved spacetimes, there is an ambiguity in the construction of a vacuum state, Fock space for quantum fields. In some cases, there may exist coordinates associated with the Killing vectors in analogy with the rectangular coordinates in Minkowski space. However, even if such privileged coordinates do exist, there are problems in the quantization of the fields [29]. That is the case of the quantum fields in BI spacetimes and the evaluation of the Bogolubov coefficients is of interest.

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