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Bianchi type I universe with viscous fluid and a Λ term: A qualitative analysis

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Abstract

The nature of cosmological solutions for a homogeneous, anisotropic Universe given by a Bianchi type-I (BI) model in the presence of a cosmological constant Λ is investigated by taking into account dissipative process due to viscosity. The system in question is thoroughly studied both analytically and numerically. It is shown that the viscosity, as well as the Λ term, exhibit essential influence on the nature of the solutions. In particular a positive Λ gives rise to an ever-expanding Universe, whereas a suitable choice of initial conditions plus a negative Λ can result in a singularity-free oscillatory mode of expansion. For some special cases it is possible to obtain oscillations in the exponential mode of expansion of the BI model even with a positive Λ, where oscillations arise by virtue of viscosity.

Keywords: Bianchi type I (BI) model; Cosmological constant; Viscous fluid

1. Introduction

In this paper we study the evolution of an anisotropic Bianchi type-I (hereafter BI) cosmological model filled with viscous fluid in the presence of a cosmological constant which is also known as the Λ term. In doing so we notice that the investigation of relativistic cosmological models usually has the energy–momentum tensor of matter generated by a perfect fluid. To consider more realistic models one must take into account the viscosity mechanisms, which have already attracted the attention of many researchers. Misner [1,2] suggested that strong dissipation due to the neutrino viscosity may considerably reduce the anisotropy of the black-body radiation. The viscosity mechanism in cosmology can explain the anomalously high entropy per baryon in the present universe [3,4]. Bulk viscosity associated with the grand-unified-theory phase transition [5] may lead to an inflationary scenario [6–8].

A uniform cosmological model filled with fluid which possesses pressure and second (bulk) viscosity was developed by Murphy [9]. The solutions that he found exhibit an interesting feature that the big bang type singularity appears in the infinite past. It should be noted that the present cosmology is based largely on Friedmann's solutions of the Einstein gravitational equations. The main feature of these solutions is their non-stationarity. Another important feature of the isotropic model is the presence of a singular point with respect to time in its space–time metric. The presence of such a singular point means that the time is restricted [10]. So the solutions obtained by Murphy with the singular point appearing in the infinite past presents definite interest. However it was soon shown that the effect obtained by Murphy is instable and vanishes if the more general models such as the anisotropic one are taken into consideration [11,12]. Exact solutions of the isotropic homogeneous cosmology for open, closed and flat universes have been found by Santos et al. [13], with the bulk viscosity being a power law of energy density.

The nature of cosmological solutions for the homogeneous Bianchi type I (BI) model was investigated by Belinskii and Khalatnikov [11] by taking into account the dissipative process due to viscosity. They showed that viscosity cannot remove the cosmological singularity but results in a qualitatively new behavior of the solutions near singularity. They found the

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remarkable property that during the time of the *big bang* matter is created by the gravitational field. BI solutions in the case of stiff matter with a shear viscosity being the power law of energy density were obtained by Banerjee [14], whereas BI models with bulk viscosity (η) that is a power law of energy density ε and when the universe is filled with stiff matter were studied by Huang [15]. The effect of bulk viscosity, with a time varying bulk viscous coefficient, on the evolution of isotropic FRW models was investigated in the context of an open thermodynamics system was studied by Desikan [16]. This study was further developed by Krori and Mukherjee [17] for anisotropic Bianchi models. Cosmological solutions with nonlinear bulk viscosity were obtained in [18]. Models with both shear and bulk viscosity were investigated in [19,20].

We studied a self-consistent system of the nonlinear spinor and/or scalar fields in a BI space–time in the presence of a perfect fluid and a Λ term [21,22] in order to clarify whether the presence of a singular point is an inherent property of the relativistic cosmological models or is it only a consequence of specific simplifying assumptions underlying these models? Recently we have considered a system of nonlinear spinor fields in a BI universe filled with viscous fluid [23]. Since the viscous fluid itself presents a growing interest, we have studied the influence of viscous fluid and Λ term in the evolution of the BI universe [24]. In that paper we consider only some special cases which allow exact solutions. In this paper along with those special cases we study some general cases, giving a qualitative analysis of the system of equations. We also perform some numerical calculations and compare the results obtained with those given in some pioneering papers in this field, e.g. [11].

2. Derivation of basic equations

Using the variational principle in this section we derive the fundamental equations for the gravitational field from the action (2.1):

$$\mathscr{S}(g;\varepsilon) = \int \mathscr{L}\sqrt{-g}\,\mathrm{d}\Omega \tag{2.1}$$

with

$$\mathscr{L} = \mathscr{L}_{\text{grav.}} + \mathscr{L}_{\text{vf}}. \tag{2.2}$$

The gravitational part of the Lagrangian (2.2) $\mathscr{L}_{\text{grav.}}$ is given by a Bianchi type-I metric, whereas the term $\mathscr{L}_{\text{vf}}$ describes a viscous fluid.

We also write the expressions for the metric functions explicitly in terms of the volume scale τ defined below (2.18). Defining the Hubble constant (2.28) in analogy with a flat Friedmann–Robertson–Walker (FRW) universe, we also derive the system of equations for τ, H and ε, with ε being the energy density of the viscous fluid, which plays the central role here.

2.1. The gravitational field

As a gravitational field we consider the Bianchi type I (BI) cosmological model. It is the simplest model of anisotropic universe that describes a homogeneous and spatially flat space–time and if filled with perfect fluid with the equation of state $p = \zeta\varepsilon, \quad \zeta < 1$, it eventually evolves into a FRW universe [25]. The isotropy of the present-day universe makes the BI model a prime candidate for studying the possible effects of an anisotropy in the early universe on modern-day data observations. In view of what has been mentioned above we choose the gravitational part of the Lagrangian (2.2) in the form

$$\mathscr{L}_{\text{grav.}} = \frac{R}{2\kappa}, \tag{2.3}$$

where R is the scalar curvature, $\kappa = 8\pi G$ being Einstein's gravitational constant. The gravitational field in our case is given by a Bianchi type I (BI) metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2\mathrm{d}x^2 - b^2\mathrm{d}y^2 - c^2\mathrm{d}z^2, \tag{2.4}$$

with a, b, c being the functions of time t only. Here the speed of light is taken to be unity.

2.2. Viscous fluid

The influence of the viscous fluid in the evolution of the Universe is performed by means of its energy–momentum tensor, which acts as the source of the corresponding gravitational field. The reason for writing $\mathscr{L}_{\text{vf}}$ in (2.2) is to underline that we are dealing with a self-consistent system. The energy–momentum tensor of a viscous field has the form

$$\begin{aligned} T_\mu^\nu &= (\varepsilon + p')u_\mu u^\nu - p'\delta_\mu^\nu + \eta g^{\nu\beta} \\ &\quad \times [u_{\mu;\beta} + u_{\beta:\mu} - u_\mu u^\alpha u_{\beta;\alpha} - u_\beta u^\alpha u_{\mu;\alpha}], \end{aligned} \tag{2.5}$$

where

$$p' = p - \left(\xi - \frac{2}{3}\eta\right) u^\mu_{;\mu}. \tag{2.6}$$

Here ε is the energy density, p is pressure, η and ξ are the coefficients of shear and bulk viscosity, respectively. Note that the bulk and shear viscosities, η and ξ, are both positively definite, i.e.,

$$\eta > 0, \quad \xi > 0. \tag{2.7}$$

They may be either constant or a function of time or energy, such as:

$$\eta = |A|\varepsilon^\alpha, \quad \xi = |B|\varepsilon^\beta. \tag{2.8}$$

The pressure p is connected to the energy density by means of a equation of state. In this report we consider the one describing a perfect fluid :

$$p = \zeta\varepsilon, \quad \zeta \in (0, 1]. \tag{2.9}$$

Note that here $\zeta \neq 0$, since for dust pressure, hence temperature is zero, that results in vanishing viscosity.

In a comoving system of reference such that $u^\mu = (1, 0, 0, 0)$ we have

$$T_0^0 = \varepsilon, \tag{2.10a}$$

$$T_1^1 = -p' + 2\eta\frac{\dot a}{a}, \tag{2.10b}$$

$$T_2^2 = -p' + 2\eta\frac{\dot{b}}{b}, \tag{2.10c}$$

$$T_3^3 = -p' + 2\eta\frac{\dot{c}}{c}. \tag{2.10d}$$

Let us introduce the dynamical scalars such as the expansion and the shear scalar as usual

$$\theta = u^\mu_{;\mu}, \quad \sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \tag{2.11}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\alpha}P^\alpha_\nu + u_{\nu;\alpha}P^\alpha_\mu\right) - \frac{1}{3}\theta P_{\mu\nu}. \tag{2.12}$$

Here P is the projection operator obeying

$$P^2 = P. \tag{2.13}$$

For the space–time with signature $(+, -, -, -)$ it has the form

$$P_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu, \quad P^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu. \tag{2.14}$$

For the BI metric the dynamical scalar has the form

$$\theta = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = \frac{\dot{\tau}}{\tau}, \tag{2.15}$$

and

$$2\sigma^2 = \frac{\dot{a}^2}{a^2} + \frac{\dot{b}^2}{b^2} + \frac{\dot{c}^2}{c^2} - \frac{1}{3}\theta^2. \tag{2.16}$$

2.3. *Field equations and their solutions*

Variation of (2.1) with respect to metric tensor $g_{\mu\nu}$ gives Einstein's field equation. On account of the Λ term for the BI space–time (2.4) this system of equations can be rewritten as

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} = \kappa T_1^1 + \Lambda, \tag{2.17a}$$

$$\frac{\ddot{c}}{c} + \frac{\ddot{a}}{a} + \frac{\dot{c}}{c}\frac{\dot{a}}{a} = \kappa T_2^2 + \Lambda, \tag{2.17b}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} = \kappa T_3^3 + \Lambda, \tag{2.17c}$$

$$\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{\dot{c}}{c}\frac{\dot{a}}{a} = \kappa T_0^0 + \Lambda, \tag{2.17d}$$

where the overdot means differentiation with respect to t and T_ν^μ is the energy–momentum tensor of a viscous fluid given above (2.10). Note that to allow a steady state cosmological solution to the gravitational field equations Einstein [26,27] introduced a fundamental constant, known as the cosmological constant or Λ term, into the system. Soon after E. Hubble had experimentally established that the Universe is expanding, Einstein returned to the original form of his equations citing his temporary modification of them as the biggest blunder of his life. The Λ term made a temporary comeback in the late 60's. Finally after the pioneering paper by Guth [8] on inflationary cosmology researchers began to study the models with Λ terms with growing interest [an excellent review on the cosmological constant can be found in [28]]. In this paper a positive Λ corresponds to the universal repulsive force, while a negative one gives an additional gravitational force. Note that a positive Λ is often considered to be one of the forms of dark energy. We would also like to note that the Λ term is also connected with the so-called *cosmic no hair* conjecture which reads: all initially expanding universes with positive cosmological constant Λ approach the de Sitter space–time asymptotically. Here we would like to mention some papers on the cosmic no hair theorem [29–37].

2.3.1. *Expressions for the metric functions*

To write the metric functions explicitly, we define a new time dependent function $\tau(t)$

$$\tau = abc = \sqrt{-g}, \tag{2.18}$$

which is indeed the volume scale of the BI space–time.

Let us now solve the Einstein equations. On account of (2.10) from (2.17a) to (2.17c) one finds the following expressions for the metric functions explicitly [24]

$$a(t) = X_1\tau^{1/3}\exp\left[(Y_1/3)\int\frac{\mathrm{e}^{-2\kappa\int\eta\mathrm{d}t}}{\tau}\mathrm{d}t\right], \tag{2.19a}$$

$$b(t) = X_2\tau^{1/3}\exp\left[(Y_2/3)\int\frac{\mathrm{e}^{-2\kappa\int\eta\mathrm{d}t}}{\tau}\mathrm{d}t\right], \tag{2.19b}$$

$$c(t) = X_3\tau^{1/3}\exp\left[(Y_3/3)\int\frac{\mathrm{e}^{-2\kappa\int\eta\mathrm{d}t}}{\tau}\mathrm{d}t\right], \tag{2.19c}$$

where the constants X_i's and Y_i's obey the following relations

$$X_1X_2X_3 = 1,$$
$$Y_1 + Y_2 + Y_3 = 0.$$

Thus, the metric functions are found explicitly in terms of τ and viscosity.

As one sees from (2.19a) to (2.19c), for $\tau = t^n$ with $n > 1$ the exponent tends to unity at large t, and the anisotropic model becomes an isotropic one.

2.3.2. *Singularity analysis*

Let us now investigate the existence of singularity (singular point) of the gravitational case, which can be done by investigating the invariant characteristics of the space–time. In general relativity these invariants are composed from the curvature tensor and the metric one. In a 4D Riemann space–time there are 14 independent invariants. Instead of analyzing all 14 invariants, one can confine this study only to 3, namely the scalar curvature $I_1 = R$, $I_2 = R_{\mu\nu}R^{\mu\nu}$, and the Kretschmann scalar $I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ [38,39]. At any regular space–time point, these three invariants I_1, I_2, I_3 should be finite. Let us rewrite these invariants in detail.

For the Bianchi I metric one finds the scalar curvature

$$I_1 = R = -2\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{\dot{c}}{c}\frac{\dot{a}}{a}\right). \tag{2.20}$$

Since the Ricci tensor for the BI metric is diagonal, the invariant $I_2 = R_{\mu\nu}R^{\mu\nu} \equiv R^{\nu}_{\mu}R^{\mu}_{\nu}$ is a sum of squares of diagonal components of a Ricci tensor, i.e.,

$$I_2 = \left[\left(R^0_0\right)^2 + \left(R^1_1\right)^2 + \left(R^2_2\right)^2 + \left(R^3_3\right)^2\right]. \quad (2.21)$$

Analogously, for the Kretschmann scalar in this case we have $I_3 = R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu}$, a sum of squared components of all nontrivial $R^{\mu\nu}{}_{\mu\nu}$, which can be written as

$$\begin{aligned} I_3 &= 4\Big[\left(R^{01}{}_{01}\right)^2 + \left(R^{02}{}_{02}\right)^2 + \left(R^{03}{}_{03}\right)^2 \\ &\quad + \left(R^{12}{}_{12}\right)^2 + \left(R^{23}{}_{23}\right)^2 + \left(R^{31}{}_{31}\right)^2\Big] \\ &= 4\left[\left(\frac{\ddot a}{a}\right)^2 + \left(\frac{\ddot b}{b}\right)^2 + \left(\frac{\ddot c}{c}\right)^2 + \left(\frac{\dot a}{a}\frac{\dot b}{b}\right)^2 + \left(\frac{\dot b}{b}\frac{\dot c}{c}\right)^2 \right.\\ &\quad \left. + \left(\frac{\dot c}{c}\frac{\dot a}{a}\right)^2\right]. \end{aligned} \quad (2.22)$$

Let us now express the foregoing invariants in terms of τ. From Eqs. (2.19) we have

$$a_i = X_i \tau^{1/3} \exp\left((Y_i/3)\int \frac{e^{-2\kappa\int\eta dt}}{\tau(t)}dt\right), \quad (2.23a)$$

$$\frac{\dot a_i}{a_i} = \frac{\dot\tau + Y_i e^{-2\kappa\int\eta dt}}{3\tau} \quad (i = 1, 2, 3,), \quad (2.23b)$$

$$\frac{\ddot a_i}{a_i} = \frac{3\tau\ddot\tau - 2\dot\tau^2 - \dot\tau Y_i e^{-2\kappa\int\eta dt} - 6\kappa\eta\tau Y_i e^{-2\kappa\int\eta dt} + Y_i^2 e^{-4\kappa\int\eta dt}}{9\tau^2}, \quad (2.23c)$$

i.e., the metric functions a, b, c and their derivatives are in functional dependence with τ. From Eqs. (2.23) one can easily verify that [24]

$$I_1 \propto \frac{1}{\tau^2}, \quad I_2 \propto \frac{1}{\tau^4}, \quad I_3 \propto \frac{1}{\tau^4}.$$

Thus we see that at any space–time point, where $\tau = 0$ the invariants I_1, I_2, and I_3 become infinity, hence the space–time becomes singular at this point.

2.4. Equations for determining τ

In the foregoing subsection we wrote the corresponding metric functions in terms of volume scale τ. In what follows, we write the equation for τ and study it in detail.

Summation of Einstein Eqs. (2.17a)–(2.17c) and 3 times (2.17d) gives

$$\ddot\tau - \frac{3}{2}\kappa\xi\dot\tau = \frac{3}{2}\kappa(\varepsilon - p)\tau + 3\Lambda\tau. \quad (2.24)$$

For the right-hand side of (2.24) to be a function of τ only, the solution to this equation is well-known [40].

The energy–momentum conservation law, i.e.,

$$T^{\nu}_{\mu;\nu} = T^{\nu}_{\mu,\nu} + \Gamma^{\nu}_{\rho\nu}T^{\rho}_{\mu} - \Gamma^{\rho}_{\mu\nu}T^{\nu}_{\rho} = 0, \quad (2.25)$$

in our case gives the following equation for ε:

$$\dot\varepsilon + \frac{\dot\tau}{\tau}\omega - \left(\xi + \frac{4}{3}\eta\right)\frac{\dot\tau^2}{\tau^2} + 4\eta(\kappa T^0_0 + \Lambda) = 0, \quad (2.26)$$

where

$$\omega = \varepsilon + p, \quad (2.27)$$

is the thermal function.

Defining a generalized Hubble constant H:

$$\frac{\dot\tau}{\tau} = \frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c} = 3H, \quad (2.28)$$

Eqs. (2.24) and (2.26) in account of (2.10) can be rewritten as

$$\dot H = \frac{\kappa}{2}(3\xi H - \omega) - (3H^2 - \kappa\varepsilon - \Lambda), \quad (2.29a)$$

$$\dot\varepsilon = 3H(3\xi H - \omega) + 4\eta(3H^2 - \kappa\varepsilon - \Lambda). \quad (2.29b)$$

In terms of dynamical scalars θ and σ the system (2.29) takes a very simple form

$$\dot\theta = \frac{3\kappa}{2}(\xi\theta - \omega) - 3\sigma^2, \quad (2.30a)$$

$$\dot\varepsilon = \theta\,(\xi\theta - \omega) + 4\eta\sigma^2. \quad (2.30b)$$

Note that Eqs. (2.30) coincide with the ones given in [14].

3. Qualitative analysis and some special solutions

In this subsection we simultaneously solve the system of equations for τ, H, and ε. It is convenient to rewrite Eqs. (2.28) and (2.29) as a single system:

$$\dot\tau = 3H\tau, \quad (3.1a)$$

$$\dot H = \frac{\kappa}{2}(3\xi H - \omega) - (3H^2 - \kappa\varepsilon - \Lambda), \quad (3.1b)$$

$$\dot\varepsilon = 3H(3\xi H - \omega) + 4\eta(3H^2 - \kappa\varepsilon - \Lambda). \quad (3.1c)$$

On account of (2.27), (2.8) and (2.9) Eqs. (3.1) now can be rewritten as

$$\dot\tau = 3H\tau, \quad (3.2a)$$

$$\dot H = \frac{\kappa}{2}(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon) - (3H^2 - \kappa\varepsilon - \Lambda), \quad (3.2b)$$

$$\dot\varepsilon = 3H(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon) + 4A\varepsilon^{\alpha}(3H^2 - \kappa\varepsilon - \Lambda). \quad (3.2c)$$

The system (3.1) have been extensively studied in the literature either partially [9,14,15] or in general [11]. In what follows, we consider the system (3.1) for some special choices of the parameters.

3.1. Qualitative analysis

Following Belinskii and Khalatnikov [11] let us now study the characters of the solutions of the dynamical system (3.1) or

(3.2). We first rewrite the system (3.1), namely (3.1b) and (3.1c) in the matrix form:

$$\begin{pmatrix} \dot{H} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} \kappa/2 & -1 \\ 3H & 4\eta \end{pmatrix} \begin{pmatrix} 3\xi H - \omega \\ 3H^2 - \kappa\varepsilon - \Lambda \end{pmatrix}. \tag{3.3}$$

Note that unlike the system studied by Belinskii and Khalatnikov the system in consideration contains a cosmological constant Λ.

3.1.1. *General properties of the system*

It is easy to note that the solutions cannot intersect the axis $\varepsilon = 0$, since $\dot{\varepsilon}|_{\varepsilon=0} = 0$, as well as the parabola

$$3H^2 - \kappa\varepsilon - \Lambda = 0, \tag{3.4}$$

as far as (3.4) is itself the integral curve. Thus, starting from the point $(H, \varepsilon) = (+\infty, 0)$, the solutions cannot enter into the "prohibited region" inside the parabola (3.4). Whether they may achieve $H < 0$ depends on the value of Λ.

3.1.2. *Critical points of the dynamical system*

(a) By virtue of linear independence of the columns of the matrix of Eq. (3.3) the critical points are the solutions of the equations

$$3\xi H - \omega = 0, \tag{3.5a}$$

$$3H^2 - \kappa\varepsilon - \Lambda = 0. \tag{3.5b}$$

i.e., they necessarily lie on the parabola (3.4). Solutions to the system (3.5) will be the roots of the equation

$$3\kappa B^2 \varepsilon^{1+2\beta} - (1+\zeta)^2\varepsilon^2 + 3\Lambda B^2 \varepsilon^{2\beta} = 0, \tag{3.6a}$$

$$H = \frac{1+\zeta}{3B}\varepsilon^{1-\beta}. \tag{3.6b}$$

The quantity of the positive roots of Eq. (3.6) according to Cartesian law is equal to the number of changes of sign of the coefficients of equations or less than that by an even number. So, for

$\Lambda > 0$ and $1/2 < \beta < 1$ (Figs. 2, 3)

or

$\Lambda < 0$ and $\beta < 1/2$ (Figs. 5, 6)

the number of roots is either 2 or zero. For the remaining cases

$\Lambda > 0$ and $\beta > 1$ (Fig. 1),
$\Lambda > 0$ and $\beta < 1/2$ (Fig. 4),
$\Lambda < 0$ and $\beta > 1/2$ (Fig. 7)

there exists only one root. The corresponding pictures of the phase curves are given in the figures cited above. The critical points are denoted by small circles. Note that here we consider the case with $\eta = 0$, i.e., $A = 0$. In the case if $\eta \neq 0$, with the increase of A the separatrix of the saddle tilts (inclines) to the left. Since the overall picture for $A \neq 0$ remains qualitatively unaltered, we only show the corresponding phase portrait for two cases, namely Fig. 8 corresponds to Fig. 1, Fig. 9 corresponds to Fig. 4. Note that for numerical calculations we

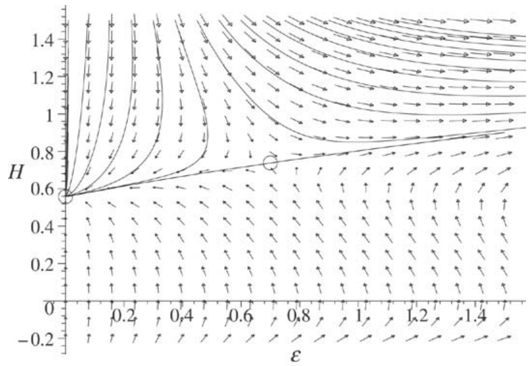

Fig. 1. Phase diagram on H–ε plane for $\beta = 1.5$, $\Lambda = .933$, $B = .720$.

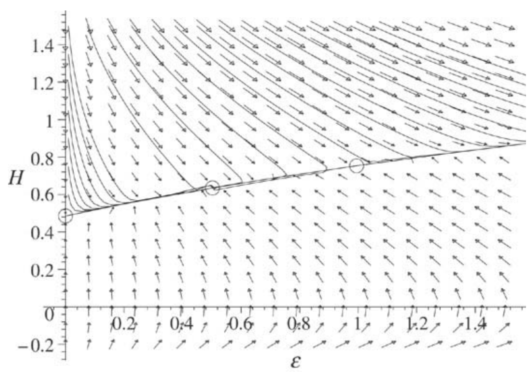

Fig. 2. Phase diagram on H–ε plane for $\beta = .75$, $\Lambda = .707$, $B = .589$.

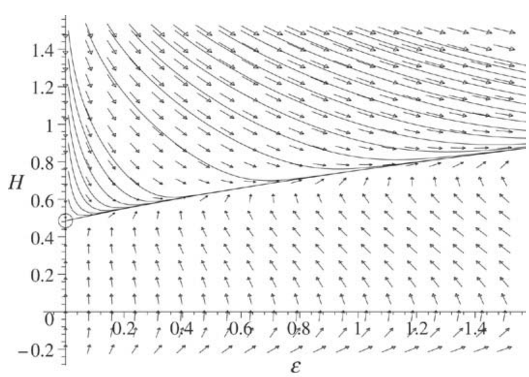

Fig. 3. Phase diagram on H–ε plane for $\beta = .75$, $\Lambda = .707$, $B = .667$.

set $\kappa = 1$, $\zeta = 0.333$ (if not mentioned otherwise). In Figs. 1–7 η is taken to be zero.

Since the equation for ε only contains η, the energy density for nontrivial η undergoes essential changes, whereas H and τ remain virtually unchanged.

The types of critical points lying on the integral curve alternate: ... saddle, attracting knot, saddle .... So it is sufficient to consider the case with maximum number of roots. Taking into account Eqs. (3.1c) and (3.4) let us now calculate

$$\lim_{\varepsilon\to+\infty} \frac{\dot{\varepsilon}}{3H\varepsilon} = \lim_{\varepsilon\to+\infty} \frac{\sqrt{3}B\varepsilon^{\beta}\sqrt{\kappa\varepsilon + \Lambda} - \varepsilon(1+\zeta)}{\varepsilon}$$

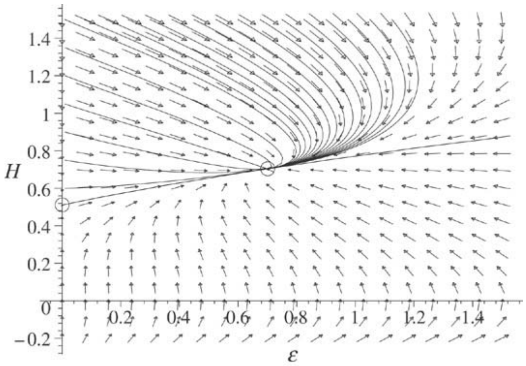

Fig. 4. Phase diagram on H–ε plane for $\beta = .05$, $\Lambda = .785$, $B = .451$.

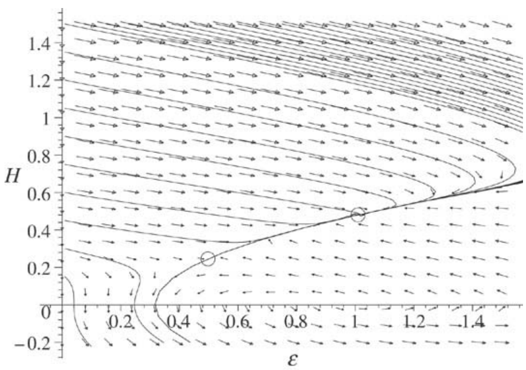

Fig. 5. Phase diagram on H–ε plane for $\beta = .05$, $\Lambda = -.317$, $B = 0.933$.

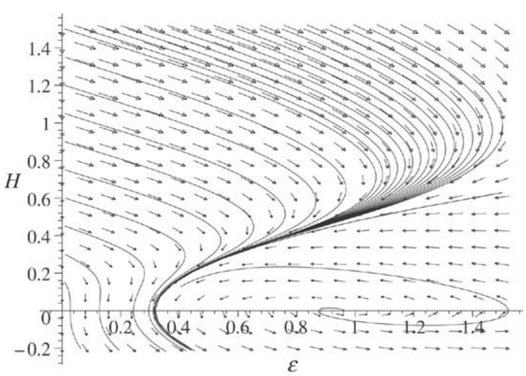

Fig. 6. Phase diagram on H–ε plane for $\beta = .05$, $\Lambda = -.317$, $B = .667$.

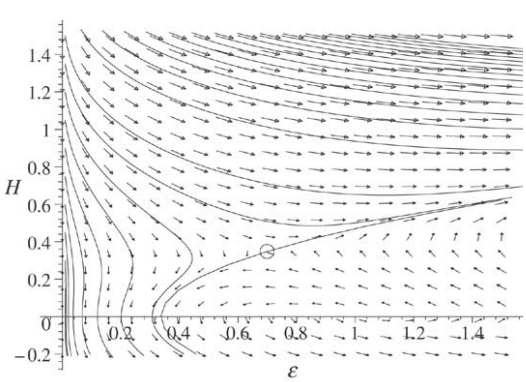

Fig. 7. Phase diagram on H–ε plane for $\beta = .75$, $\Lambda = -.337$, $B = 1.169$.

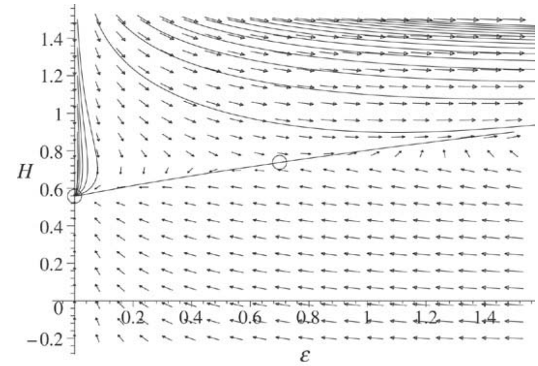

Fig. 8. Phase diagram on H–ε plane for $\beta = 1.5$, $\Lambda = .933$, $B = .720$, $A = 1$, $\alpha = 1$.

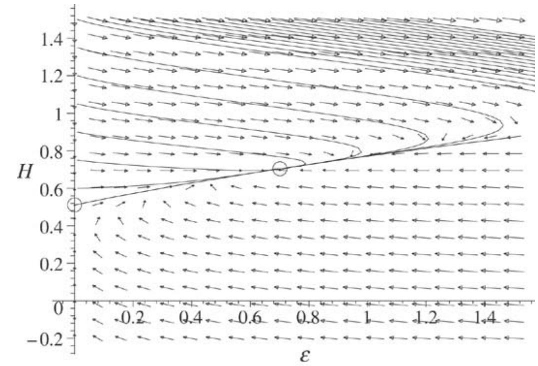

Fig. 9. Phase diagram on H–ε plane for $\beta = .05$, $\Lambda = .785$, $B = .451$, $A = 1$, $\alpha = 1$, $\kappa = 1$.

$$
\begin{aligned}
&= \sqrt{3}B\sqrt{\kappa\varepsilon^{(2\beta-1)} + \Lambda\varepsilon^{-2}} - (1+\zeta) \\
&= \begin{cases} -(1+\zeta) < 0, & \beta < 1/2, \\ B\sqrt{3\kappa} - (1+\zeta), & \beta = 1/2, \\ +\infty > 0, & \beta > 1/2. \end{cases}
\end{aligned}
\tag{3.7}
$$

So, the latest critical point for $\beta < 1/2$ is attracting knot and for $\beta > 1/2$ is saddle. In case of $\beta = 1/2$ we have saddle if $B\sqrt{3\kappa} - (1+\zeta) > 0$ and attracting knot otherwise.

(b) It is obvious that if $\Lambda \geq 0$ the points of intersection of the boundary are the critical points

$$H = \pm\sqrt{\Lambda/3}, \tag{3.8a}$$

$$\varepsilon = 0. \tag{3.8b}$$

(c) For $H < 0$ there may exist critical points, if the columns of the matrix of (3.3) are linearly dependent. In that case the critical points are the roots of the equation

$$3\kappa(\zeta - 1)\varepsilon + 6\kappa^2 AB\varepsilon^{\alpha+\beta} + 8\kappa^2 A^2\varepsilon^{2\alpha} - 6\Lambda = 0, \tag{3.9}$$

and

$$H = -\frac{2}{3}\kappa A\varepsilon^{\alpha}. \tag{3.10}$$

In the case of $\eta = 0$ the roots of the characteristic equation

$$\left|\frac{D(\dot{H}, \dot{\varepsilon})}{D(H, \varepsilon)} - \mu\right| = 0, \tag{3.11}$$

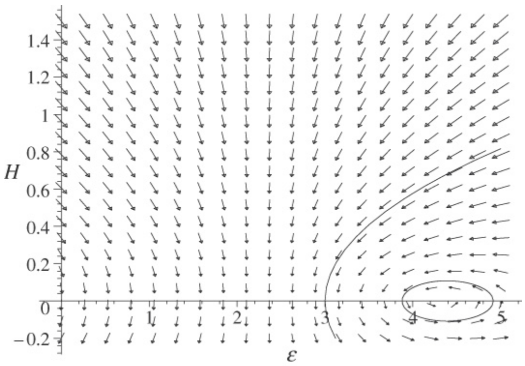

Fig. 10. Phase diagram on H–ε plane for $\Lambda = -3, \zeta = 0.333, C_2 = 1, C_3 = 1$.

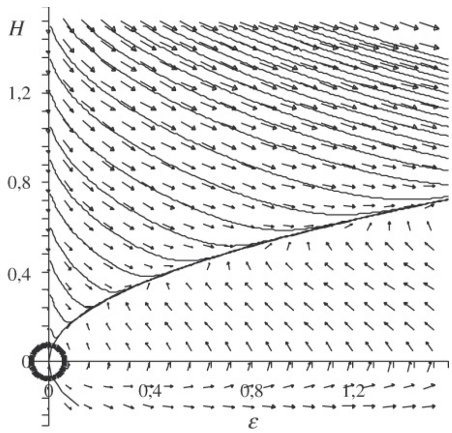

Fig. 11. Phase diagram on H–ε plane for $\beta = 0.5, \Lambda = 0, B = 0.589, A = 0$,

are

$$\mu_{1,2} = \frac{3\kappa\xi \pm \sqrt{9\kappa^2\xi^2 + 48\Lambda(1+\zeta)}}{4}. \tag{3.12}$$

The critical point $(H, \varepsilon) = (0, 2\Lambda/[\kappa(\zeta - 1)])$ is of type divergent focus if $\Lambda > -9\kappa^2\xi^2/[48(1+\zeta)]$ or divergent knot if $\Lambda < -9\kappa^2\xi^2/[48(1+\zeta)]$.

In the cases illustrated in Figs. 5 and 7, $H \to \infty$ and $\varepsilon \to \infty$ as $t \to \infty$, whereas for the cases given in Fig. 6 one observes increasing oscillation bounded by the attracting parabola (3.4).

3.1.3. Integral curves

For $\Lambda \geq 0$ the solutions starting from the upper half-plane $H > 0$ cannot enter into the lower one. For $\Lambda < 0$ some of the solutions may enter into the lower half-plane through the segment $H = 0$ and $\Lambda \leq 0 \leq \varepsilon$ and never returns back, since $\dot{H}|_{H=0} < 0$.

3.2. Numerical solutions

In this subsection solutions to the system of equations (3.1) has been obtained numerically. Evolution of the Hubble constant H, energy density ε and volume scale τ corresponding to the cases studied above with different B, β and Λ has been illustrated in Figs. 1–15. As one sees, for a positive Λ the volume scale τ expands exponentially, whereas, for a

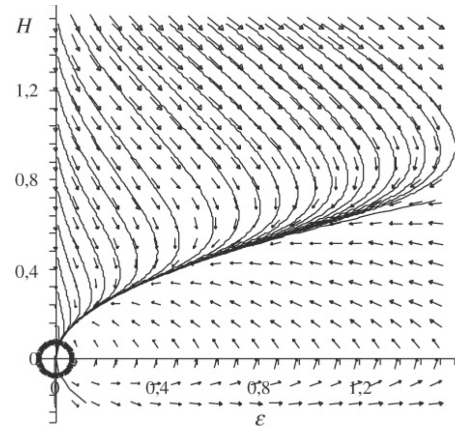

Fig. 12. Phase diagram on H–ε plane for $\beta = 0.5, \Lambda = 0, B = 0.856$.

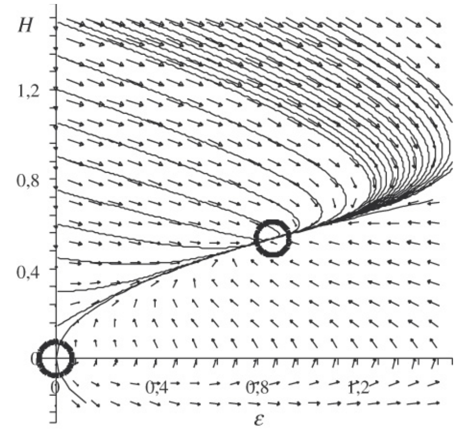

Fig. 13. Phase diagram on H–ε plane for $\beta = 0.05, \Lambda = 0, B = 0.71, A = 0$.

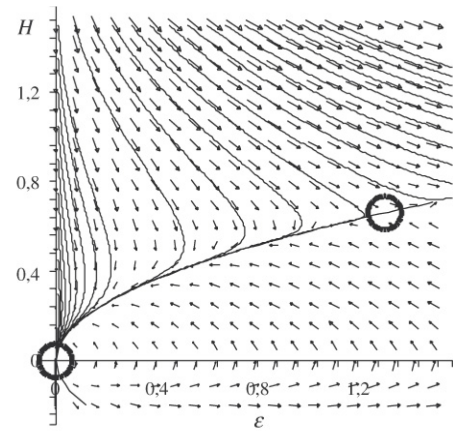

Fig. 14. Phase diagram on H–ε plane for $\beta = 0.75, \Lambda = 0, B = 0.71, A = 0$.

negative Λ there exist solutions where τ initially expands and after reaching some maximum begins to contract and finally collapses into a point, thus giving rise to space–time singularity. Beside this, as one sees from Fig. 15, a suitable choice of initial conditions gives rise to a singularity-free oscillatory mode of expansion of the Universe.

Table 1
Evolution of BI universe depending on the parameters

	$0 < \beta < 0.5$	$\beta = 0.5$		$0.5 < \beta < 1.0$	$\beta = 1$		$\beta > 1$
		$\varphi_1 < \varphi_3$	$\varphi_1 > \varphi_3$		$\varphi_2 < \varphi_3$	$\varphi_2 > \varphi_3$	
$\Lambda < 0$	Figs. 5, 6	Fig. 6			Fig. 7		
$\Lambda = 0$	Fig. 13	Fig. 12	Fig. 11		Fig. 14		
$\Lambda > 0$	Fig. 4		Fig. 3	Figs. 2, 3	Fig. 1	Fig. 3	Fig. 1

Here we use the notations $\varphi_1 := 3\kappa B^2$, $\varphi_2 := 3\Lambda B^2$ and $\varphi_3 := (1+\zeta)^2$.

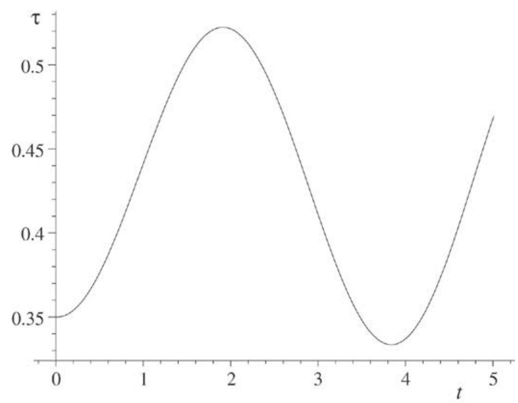

Fig. 15. Evolution of the BI universe corresponding to the phase diagram given in Fig. 10. As one sees, the BI universe in this case undergoes an oscillatory mode of expansion.

3.3. Exact solutions

In this subsection we consider some special cases allowing exact solutions. Since the system was thoroughly studied by one of the authors (B.S.) in a recent paper [24], here we only point out the main results leaving the details aside. Note that in [24] the Λ term has the opposite sign, i.e., in that case positive Λ corresponds to an additional gravitational force, while the negative one acts as a repulsive force. In this paper we use the conventional sign.

3.3.1. Case with bulk viscosity

In this case Eq. (3.1c) takes the form

$$\dot{\varepsilon} = 3H(3\xi H - \omega). \tag{3.13}$$

In view of (3.13) the system (3.1) admits the following first integral

$$\tau^2(\kappa\varepsilon - 3H^2 + \Lambda) = C_1, \quad C_1 = \text{const.} \tag{3.14}$$

It can be shown that in the presence of a positive Λ the evolution of the Universe never comes to a halt, it either expands further or begin to contract depending on the sign of $H = \pm\sqrt{\Lambda/3}$, $\Lambda > 0$. Taking into account that $3H = \dot{\tau}/\tau$ in this case we find $\tau \sim \exp[\pm\sqrt{3\Lambda}\,t]$. Choosing the positive root we see that the initially expanding Universe approaches de Sitter space–time asymptotically, i.e., in this case the cosmic no hair conjecture takes place.

If the bulk viscosity is chosen to be inversely proportional to expansion, i.e., $\xi\theta = C_2$ where C_2 is some constant Eq. (2.24) admits the solution in quadrature:

$$\int \frac{\mathrm{d}\tau}{\sqrt{C_2^2 + C_0^0\tau^2 + C_1^1\tau^{1-\zeta}}} = t + t_0, \tag{3.15}$$

where C_2^2 and t_0 are some constants. Here, $C_0^0 = 3\kappa C_2/(1+\zeta) + 3\Lambda$ and $C_1^1 = 3\kappa C_3/(1+\zeta)$. It can be shown that a suitable choice of C_2^2 and τ_0 (the initial value of τ) can give rise to an oscillatory mode of expansion with τ being always positive, i.e., a singularity free evolution of the Universe. The phase portrait of the $(H,\ \varepsilon)$ plane and the evolution of the BI universe corresponding to this portrait allowing oscillatory solutions are given in Figs. 10 and 15. If the value of ζ is taken to be unity which corresponds to a stiff matter the BI universe first expands, reaches its maximum and then contracts into a point, thus giving rise to space–time singularity. The classification of the type of evolution of the BI universe depending on the problem parameters are given in Table 1.

3.3.2. Case with shear and bulk viscosity

Consider the case when the bulk viscosity ξ is a constant and the shear viscosity η is proportional to the expansion, i.e.,

$$\xi = \text{const.}, \quad \eta \propto \theta = 3H. \tag{3.16}$$

In this case the system allows non-periodic or exponential expansion of τ both for positive and negative Λ [24].

4. Conclusion

We investigated the cosmological solutions to the equations of General Relativity for the homogeneous anisotropic Bianchi type I model by taking into account dissipative processes due to viscosity and cosmological constant (Λ term). A detailed analysis showed that the viscosity, as well as the Λ term, exhibit an essential influence on the nature of the solutions. The classification of the solutions was pursued for the viscosity being some power law of energy density, namely, $\eta = A\varepsilon^{\alpha}$ and $\xi = B\varepsilon^{\beta}$. It was noticed that for $\Lambda > 0$ the Universe expands forever with a logarithmic velocity H, which, depending on the viscosity either becomes constant or increases infinitely. In the process behavior of the energy density ε is analogous to that of H except in the case when $\varepsilon \to 0$. For $\Lambda < 0$, beside the variants mentioned above, there exists a few other possibilities depending on some special choices: contraction of the Universe into a point, thus giving rise to a space–time singularity; a regime of increasing oscillation by virtue of suitable initial conditions; or a singularity-free oscillatory mode of expansion.

References

[1] W. Misner, Transport process in the primordial fireball, Nature 214 (1967) 40–41.
[2] W. Misner, The isotropy of the universe, Astrophys. J. 151 (1968) 431–457.
[3] S. Weinberg, Entropy generation and the survival of protogalaxies in an expanding Universe, Astrophys. J. 168 (2) (1972) 175–194.
[4] S. Weinberg, Gravitation and Cosmology, Wiley, New York, 1972.
[5] P. Langacker, Grand unified theories and proton decay, Phys. Rep. 72 (4) (1981) 185–385.
[6] L. Waga, R.C. Falcan, R. Chanda, Bulk-viscosity-driven inflationary model, Phys. Rev. D 33 (1986) 1839–1841.
[7] T. Pacher, J.A. Stein-Schabas, M.S. Turner, Can bulk viscosity drive inflation? Phys. Rev. D 36 (1987) 1603–1606.
[8] A. Guth, Inflationary universe: A possible solution to the horizon and flatness problems, Phys. Rev. D 23 (2) (1981) 347–356.
[9] G.L. Murphy, Big-Bang model without singularities, Phys. Rev. D 8 (1973) 4231–4233.
[10] V.A. Belinskii, I.M. Khalatnikov, E.M. Lifshitz, Oscillatory approach to a singular point in the relativistic cosmology, Adv. Phys. 19 (1970) 525–573.
[11] V.A. Belinskii, I.M. Khalatnikov, On the effect of viscosity on the nature of cosmological evolution, Sov. J. JETP 69 (2(8)) (1975) 401–413.
[12] J.D. Barrow, String-driven inflationary and deflationary cosmological models, Nuclear Phys. B 310 (3–4) (1988) 743–763.
[13] N.O. Santos, R.S. Dias, A. Banerjee, Isotropic homogeneous universe with viscous fluid, J. Math. Phys. 26 (1985) 876–881.
[14] A. Banerjee, S.B. Duttachoudhury, A.K. Sanyal, Bianchi type I cosmological model with a viscous fluid, J. Math. Phys. 26 (1985) 3010–3015.
[15] W. Huang, Anisotropic cosmological models with energy density dependent bulk viscosity, J. Math. Phys. 31 (6) (1990) 1456–1462.
[16] K. Desikan, Cosmological models with bulk viscosity in the presence of particle creation, Gen. Relativity Gravitation 29 (1997) 435–443.
[17] K.D. Krori, A. Mukherjee, Bianchi cosmologies with particle creation and bulk viscosity, Gen. Relativity Gravitation 32 (2000) 1429–1438.
[18] L.P. Chimento, A.S. Jacubi, V. Mèndez, R. Maartens, Cosmological solutions with nonlinear bulk viscosity, Classical Quant. Grav. 14 (1997) 3363–3375.
[19] H. van Elst, P. Dunsby, R. Tavakol, Constraints on inflationary solutions in the presence of shear and bulk viscosity, Gen. Relativity Gravitation 27 (1995) 171–191.
[20] V.R. Gavrilov, V.N. Melnikov, R. Triay, Exact solutions in multidimensional cosmology with shear and bulk viscosity, Classical Quant. Grav. 14 (1997) 2203–2218.
[21] B. Saha, Spinor field in Bianchi type-I universe: regular solutions, Phys. Rev. D 64 (2001) 123501.
[22] B. Saha, T. Boyadjiev, Bianchi type-I cosmology with scalar and spinor fields, Phys. Rev. D 69 (2004) 124010.
[23] B. Saha, Nonlinear spinor field in Bianchi type-I universe filled with viscous fluid: some special solutions, Romanian Rep. Phys. 57 (1) (2005) 7–24.
[24] B. Saha, Bianchi type I universe with viscous fluid, Modern Phys. Lett. A 20 (28) (2005) 2127–2143.
[25] K.C. Jacobs, Spatially homogeneous and euclidean cosmological models with shear, Astrophys. J. 153 (2) (1968) 661–678.
[26] A. Einstein, Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie, Sitzungsber, Preuss. Acad. Wiss. 1 (1917) 142–152.
[27] A. Einstein, Spielen die gravitationsfelder im aufbau der materiellen elementarteilchen eine wesentliche rolle? Sitzungsber, Preuss. Acad. Wiss. 1 (1919) 349–356.
[28] T. Padmanabhan, Cosmological constant — the weight of the vacuum, Phys. Rep. 380 (5–6) (2003) 235–320.
[29] S.W. Hawking, I.G. Moss, Supercooled phase transitions in the very early universe, Phys. Lett. B 110 (1) (1982) 35–38.
[30] R.W. Wald, Asymptotic behavior of homogeneous cosmological models in the presence of a positive cosmological constant, Phys. Rev. D. 28 (8) (1983) 2118–2120.
[31] M. Bruni, S. Matarrese, O. Pantano, A local view of the observable universe, Phys. Rev. Lett. 74 (11) (1995) 1916–1919.
[32] H.-a. Shinkai, K.-i. Maeda, Can gravitational waves prevent inflation? Phys. Rev. D 48 (8) (1993) 3910–3914.
[33] H.-a. Shinkai, K.-i. Maeda, Cosmic no-hair theorem in homogeneous spacetimes: I. Bianchi models, Classical Quant. Grav. 10 (1993) 703–734.
[34] H.-a. Shinkai, K.-i. Maeda, Generality of inflation in a planar universe, Phys. Rev. D 49 (12) (1994) 6367–6383.
[35] J.E. Lidsey, D. Serry, Cosmic no hair for braneworlds with a bulk dilaton field, Phys. Rev. D 72 (2005) 103507.
[36] J.E. Lidsey, Cosmic no hair for collapsing universes, Classical Quant. Grav. 23 (2006) 3517–3530.
[37] B.C. Paul, Inflation in Bianchi models and the cosmic no hair theorem in a brane world, Phys. Rev. D 66 (2002) 124019.
[38] K.A. Bronnikov, G.N. Shikin, Cylindrically symmetric solitons with nonlinear self-gravitating scalar fields, Gravit. Cosmol. 7 (2001) 231–240.
[39] S. Fay, Sufficient conditions for curvature invariants to avoid divergences in hyperextended scalar–tensor theory for Bianchi models, Classical Quant. Grav. 17 (2000) 2663–2673.
[40] E. Kamke, Differentialgleichungen losungsmethoden und losungen, Akademische Verlagsgesellschaft, Leipzig, 1957.