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NONLINEAR SPINOR FIELD IN BIANCHI TYPE-I UNIVERSE FILLED WITH VISCOUS FLUID: SOME SPECIAL SOLUTIONS

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Abstract. We consider a system of nonlinear spinor and a Bianchi type I gravitational fields in presence of viscous fluid. The nonlinear term in the spinor field Lagrangian is chosen to be λF, with λ being the self-coupling constant and F being a function of the invariants I an J constructed from bilinear spinor forms S and P. We consider the cases when F is the power law of its arguments. Self-consistent solutions to the spinor and BI gravitational field equations are obtained in terms of τ, where τ is the volume scale of BI universe. System of equations for τ and ε, where ε is the energy of the viscous fluid, is deduced. This system is solved for some special cases.

Key words: spinor field, Bianchi type I (BI) model, cosmological constant.

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I. INTRODUCTION

The investigation of relativistic cosmological models usually has the energy momentum tensor of matter generated by a perfect fluid. To consider more realistic models one must take into account the viscosity mechanisms, which have already attracted the attention of many researchers. Misner [1, 2] suggested that strong dissipative due to the neutrino viscosity may considerably reduce the anisotropy of the black-body radiation. Viscosity mechanism in cosmology can explain the anomalously high entropy per baryon in the present universe [3, 4]. Bulk viscosity associated with the grand-unified-theory phase transition [5] may lead to an inflationary scenario [6, 7, 8].

A uniform cosmological model filled with fluid which possesses pressure and second (bulk) viscosity was developed by Murphy [9]. The solutions that he found

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exhibit an interesting feature that the big bang type singularity appears in the infinite past. Exact solutions of the isotropic homogeneous cosmology for open, closed and flat universe have been found by Santos *et al.* [10], with the bulk viscosity being a power function of energy density.

The nature of cosmological solutions for homogeneous Bianchi type I (BI) model was investigated by Belinsky and Khalatnikov [11] by taking into account dissipative process due to viscosity. They showed that viscosity cannot remove the cosmological singularity but results in a qualitatively new behavior of the solutions near singularity. They found the remarkable property that during the time of the *big bang* matter is created by the gravitational field. BI solutions in case of stiff matter with a shear viscosity being the power function of of energy density were obtained by Banerjee [12], whereas BI models with bulk viscosity (η) that is a power function of energy density ε and when the universe is filled with stiff matter were studied by Huang [13]. The effect of bulk viscosity, with a time varying bulk viscous coefficient, on the evolution of isotropic FRW models investigated in the context of open thermodynamics system was studied by Desikan [14]. This study was further developed by Krori and Mukherjee [15] for anisotropic Bianchi models. Cosmological solutions with nonlinear bulk viscosity were obtained in [16]. Models with both shear and bulk viscosity were investigated in [17, 18].

Though Murphy [9] claimed that the introduction of bulk viscosity can avoid the initial singularity at finite past, results obtained in [19] show that, it is, in general, not valid, since for some cases big bang singularity occurs in finite past. To eliminate the initial singularities a self-consistent system of nonlinear spinor and BI gravitational field was considered by us in a series of papers [20, 21, 22, 23]. For some cases we were able to find field (both matter and gravitational) configurations that were always regular. In the papers mentioned above we considered the system of interacting nonlinear spinor and/or scalar fields in a BI universe filled with perfect fluid. We also study the above system in the presence of the cosmological constant Λ (both constant and time varying [23]). A nonlinear spinor field, suggested by the symmetric coupling between nucleons, muons, and leptons, has been investigated by Finkelstein *et al.* [24] in the classical approximation. Although the existence of spin-1/2 fermion is both theoretically and experimentally undisputed, these are described by *quantum* spinor fields. Possible justifications for the existence of classical spinors has been addressed in [25]. In view of what has been mentioned above, it would be interesting to study the influence of viscous fluid to the system of material (say spinor and/or scalar) and BI gravitational fields in the presence of a Λ-term as well. In this report we study the system of nonlinear spinor field in a BI universe filled with viscous fluid. We write the corresponding system of equations in general and present some solutions for some special cases. We plan to study the system for more general cases in some of our future papers.

II. DERIVATION OF BASIC EQUATIONS

In this section we derive the fundamental equations for the interacting spinor, scalar and gravitational fields from the action and write their solutions in term of the volume scale τ defined below (2.23). We also derive the equation for τ which plays the central role here.

We consider a system of nonlinear spinor, scalar and BI gravitational field in the presence of perfect fluid given by the action

$$\mathcal{S}(g;\psi,\bar{\psi}) = \int \mathcal{L}\sqrt{-g}\,\mathrm{d}\Omega \tag{2.1}$$

with

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{sp} + \mathcal{L}_m. \tag{2.2}$$

The gravitational part of the Lagrangian (2.2) is given by a Bianchi type I (BI hereafter) space-time, whereas the remaining parts are the usual spinor field Lagrangian with a self-coupling and a viscous fluid as well.

A. MATERIAL FIELD LAGRANGIAN

For a spinor field ψ, symmetry between ψ and $\bar{\psi}$ appears to demand that one should choose the symmetrized Lagrangian [26]. Keep it in mind we choose the spinor field Lagrangian as

$$\mathcal{L}_{sp} = \frac{i}{2}[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi] - m\bar{\psi}\psi + \lambda F, \tag{2.3}$$

Here m is the spinor mass, λ is the self-coupling constant and F is some arbitrary functions of invariants generated from the real bilinear forms of a spinor field with the form

$$S = \bar{\psi}\psi \qquad \text{(scalar)}, \tag{2.4a}$$

$$P = i\bar{\psi}\gamma^5\psi \qquad \text{(pseudoscalar)}, \tag{2.4b}$$

$$v^{\mu} = (\bar{\psi}\gamma^{\mu}\psi) \qquad \text{(vector)}, \tag{2.4.c}$$

$$A^{\mu} = (\bar{\psi}\gamma^5\gamma^{\mu}\psi) \qquad \text{(pseudovector)}, \tag{2.4.d}$$

$$Q^{\mu\nu} = (\bar{\psi}\sigma^{\mu\nu}\psi) \qquad \text{(antisymmetric tensor)}, \tag{2.4.e}$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}]$. Invariants, corresponding to the bilinear forms, are

$$I = S^2, \tag{2.5.a}$$

$$J = P^2, \tag{2.5.b}$$

$$I_v = v_\mu\, v^\mu = (\bar\psi\gamma^\mu\psi)\, g_{\mu\nu}\, (\bar\psi\gamma^\nu\psi), \tag{2.5.c}$$

$$I_A = A_\mu\, A^\mu = (\bar\psi\gamma^5\gamma^\mu\psi)\, g_{\mu\nu}\, (\bar\psi\gamma^5\gamma^\nu\psi), \tag{2.5.d}$$

$$I_Q = Q_{\mu\nu}\, Q^{\mu\nu} = (\bar\psi\sigma^{\mu\nu}\psi)\, g_{\mu\alpha} g_{\nu\beta}\, (\bar\psi\sigma^{\alpha\beta}\psi). \tag{2.5.e}$$

According to the Pauli-Fierz theorem [27] among the five invariants only I and J are independent as all the other can be expressed by them: $I_V = -I_A = I + J$ and $I_Q = I - J$. Therefore, we choose $F = F(I, J)$, thus claiming that it describes the nonlinearity in the most general of its form [21]. Note that setting $\lambda = 0$ in (2.3) we come to the case with linear spinor field.

The term $\mathscr{L}_m$ describes the Lagrangian density of viscous fluid.

B. THE GRAVITATIONAL FIELD

As a gravitational field we consider the Bianchi type I (BI) cosmological model. It is the simplest model of anisotropic universe that describes a homogeneous and spatially flat space-time and if filled with perfect fluid with the equation of state $p = \zeta\varepsilon$, $\zeta < 1$, it eventually evolves into a FRW universe [28]. The isotropy of the present-day universe makes BI model a prime candidate for studying the possible effects of an anisotropy in the early universe on modern-day data observations. In view of what has been mentioned above we choose the gravitational part of the Lagrangian (2.2) in the form

$$\mathscr{L}_g = \frac{R}{2\kappa}, \tag{2.6}$$

where R is the scalar curvature, $\kappa = 8\pi G$ being Einstein's gravitational constant. The gravitational field in our case is given by a Bianchi type I (BI) metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2\mathrm{d}x^2 - b^2\mathrm{d}y^2 - c^2\mathrm{d}z^2, \tag{2.7}$$

with a, b, c being the functions of time t only. Here the speed of light is taken to be unity.

The metric (2.7) has the following non-trivial Christoffel symbols

$$\begin{aligned} &\Gamma^1_{10} = \frac{\dot a}{a}, \quad \Gamma^2_{20} = \frac{\dot b}{b}, \quad \Gamma^3_{30} = \frac{\dot c}{c} \\ &\Gamma^0_{11} = a\dot a, \quad \Gamma^0_{22} = b\dot b, \quad \Gamma^0_{33} = c\dot c. \end{aligned} \tag{2.8}$$

The nontrivial components of the Ricci tensors are

$$R_0^0 = -\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}\right), \tag{2.9a}$$

$$R_1^1 = -\left[\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right)\right], \tag{2.9b}$$

$$R_2^2 = -\left[\frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\left(\frac{\dot{c}}{c} + \frac{\dot{a}}{a}\right)\right], \tag{2.9c}$$

$$R_3^3 = -\left[\frac{\ddot{c}}{c} + \frac{\dot{c}}{c}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right)\right]. \tag{2.9d}$$

From (2.9) one finds the following Ricci scalar for the BI universe

$$R = -2\left(\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{\dot{c}}{c}\frac{\dot{a}}{a}\right). \tag{2.10}$$

The non-trivial components of Riemann tensors in this case read

$$\begin{aligned} R_{01}^{01} &= \frac{\ddot{a}}{a}, \quad R_{02}^{02} = \frac{\ddot{b}}{b}, \quad R_{03}^{03} = \frac{\ddot{c}}{c}, \\ R_{12}^{12} &= -\frac{\dot{a}}{a}\frac{\dot{b}}{b}, \quad R_{23}^{23} = -\frac{\dot{b}}{b}\frac{\dot{c}}{c}, \quad R_{31}^{31} = -\frac{\dot{c}}{c}\frac{\dot{a}}{a}. \end{aligned} \tag{2.11}$$

Now having all the non-trivial components of Ricci and Riemann tensors, one can easily write the invariants of gravitational field which we need to study the space-time singularity. We return to this study at the end of this section.

C. FIELD EQUATIONS

Let us now write the field equations corresponding to the action (2.1).

Variation of (2.1) with respect to spinor field $\psi(\bar{\psi})$ gives spinor field equations

$$i\gamma^\mu \nabla_\mu \psi - m\psi + \mathcal{D}\psi + \mathcal{G} i\gamma^5 \psi = 0, \tag{2.12a}$$

$$i\nabla_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} - \mathcal{D}\bar{\psi} - \mathcal{G} i\bar{\psi}\gamma^5 = 0, \tag{2.12b}$$

where we denote

$$\mathcal{D} = 2\lambda S \frac{\partial F}{\partial I}, \quad \mathcal{G} = 2\lambda P \frac{\partial F}{\partial J}.$$

Variation of (2.1) with respect to metric tensor $g_{\mu\nu}$ which gives Einstein's field equation in account of the Λ-term has the form

$$G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = \kappa T_\mu^\nu - \delta_\mu^\nu \Lambda. \tag{2.13}$$

In view of (2.9) and (2.10) for the BI space-time (2.7) we rewrite the Eq. (2.13) as

$$\frac{\ddot b}{b}+\frac{\ddot c}{c}+\frac{\dot b}{b}\frac{\dot c}{c}=\kappa T_1^1-\Lambda, \tag{2.14a}$$

$$\frac{\ddot c}{c}+\frac{\ddot a}{a}+\frac{\dot c}{c}\frac{\dot a}{a}=\kappa T_2^2-\Lambda, \tag{2.14b}$$

$$\frac{\ddot a}{a}+\frac{\ddot b}{b}+\frac{\dot a}{a}\frac{\dot b}{b}=\kappa T_3^3-\Lambda, \tag{2.14c}$$

$$\frac{\dot a}{a}\frac{\dot b}{b}+\frac{\dot b}{b}\frac{\dot c}{c}+\frac{\dot c}{c}\frac{\dot a}{a}=\kappa T_0^0-\Lambda, \tag{2.14d}$$

where over dot means differentiation with respect to t and T_ν^μ is the energy-momentum tensor of the material field given by

$$T_\mu^\nu=T_{\text{sp}\,\mu}^{\ \ \nu}+T_{\text{m}\,\mu}^{\ \ \nu}. \tag{2.15}$$

Here $T_{\text{sp}\,\mu}^{\ \ \nu}$ is the energy-momentum tensor of the spinor field which with regard to (2.12) has the form

$$\begin{aligned}T_{\text{sp}\,\mu}^{\ \ \rho}=\frac{i}{4}g^{\rho\nu}(\bar\psi\gamma_\mu\nabla_\nu\psi+\bar\psi\gamma_\nu\nabla_\mu\psi-\nabla_\mu\bar\psi\gamma_\nu\psi-\nabla_\nu\bar\psi\gamma_\mu\psi)+\\ +\delta_\mu^\rho(\mathcal{D}S+\mathcal{G}P-\lambda F(I,J)).\end{aligned} \tag{2.16}$$

$T_{\mu(\text{m})}^\nu$ is the energy-momentum tensor of a viscous fluid having the form

$$T_{\mu(\text{m})}^\nu=(\varepsilon+p')u_\mu u^\nu-p'\delta_\mu^\nu+\eta g^{\nu\beta}[u_{\mu;\beta}+u_{\beta:\mu}-u_\mu u^\alpha u_{\beta;\alpha}-u_\beta u^\alpha u_{\mu;\alpha}], \tag{2.17}$$

where

$$p'=p-\left(\xi-\frac{2}{3}\eta\right)u_{;\mu}^\mu. \tag{2.18}$$

Here ε is the energy density, p – pressure, η and ξ are the coefficients of shear and bulk viscosity, respectively. In a comoving system of reference such that $u^\mu=(1,0,0,0)$ we have

$$T_{0(\text{m})}^0=\varepsilon, \tag{2.19a}$$

$$T_{1(\text{m})}^1=-p'+2\eta\frac{\dot a}{a}, \tag{2.19b}$$

$$T_{2(\text{m})}^2=-p'+2\eta\frac{\dot b}{b}, \tag{2.19c}$$

$$T_{3(\text{m})}^3=-p'+2\eta\frac{\dot c}{c}. \tag{2.19d}$$

In the Eqs. (2.12) and (2.16) ∇_μ is the covariant derivatives acting on a spinor field as [29,30]

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \quad \nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial x^\mu} + \bar{\psi}\Gamma_\mu, \tag{2.20}$$

where Γ_μ are the Fock-Ivanenko spinor connection coefficients defined by

$$\Gamma_\mu = \frac{1}{4}\gamma^\sigma (\Gamma^\nu_{\mu\sigma}\gamma_\nu - \partial_\mu \gamma_\sigma). \tag{2.21}$$

For the metric (2.7) one has the following components of the spinor connection coefficients

$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{1}{2}\dot{a}(t)\bar{\gamma}^1\bar{\gamma}^0, \quad \Gamma_2 = \frac{1}{2}\dot{b}(t)\bar{\gamma}^2\bar{\gamma}^0, \quad \Gamma_3 = \frac{1}{2}\dot{c}(t)\bar{\gamma}^3\bar{\gamma}^0. \tag{2.22}$$

The Dirac matrices $\gamma^\mu(x)$ of curved space-time are connected with those of the Minkowski one as follows:

$$\gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \bar{\gamma}^1/a, \quad \gamma^2 = \bar{\gamma}^2/b, \quad \gamma^3 = \bar{\gamma}^3/c$$

with

$$\bar{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \bar{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \bar{\gamma}^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix},$$

where σ_i are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the $\bar{\gamma}$ and the σ matrices obey the following properties:

$$\begin{aligned}
&\bar{\gamma}^i\bar{\gamma}^j + \bar{\gamma}^j\bar{\gamma}^i = 2\eta^{ij}, \quad i,j = 0,1,2,3 \\
&\bar{\gamma}^i\bar{\gamma}^5 + \bar{\gamma}^5\bar{\gamma}^i = 0, \quad (\bar{\gamma}^5)^2 = I, \quad i = 0,1,2,3 \\
&\sigma^j\sigma^k = \delta_{jk} + i\varepsilon_{jkl}\sigma^l, \quad j,k,l = 1,2,3
\end{aligned}$$

where $\eta_{ij} = \{1,-1,-1,-1\}$ is the diagonal matrix, δ_{jk} is the Kronekar symbol and ε_{jkl} is the totally antisymmetric matrix with $\varepsilon_{123} = +1$.

We study the space-independent solutions to the spinor field equations (2.12) so that $\psi = \psi(t)$. Here we define

$$\tau = abc = \sqrt{-g} \tag{2.23}$$

The spinor field equation (2.12a) in account of (2.20) and (2.22) takes the form

$$i\bar{\gamma}^0\left(\frac{\partial}{\partial t}+\frac{\dot{\tau}}{2\tau}\right)\psi - m\psi + \mathscr{D}\psi + \mathscr{G}i\gamma^5\psi = 0. \tag{2.24}$$

Setting $V_j(t)=\sqrt{\tau}\psi_j(t),\ j=1, 2,\ 3, 4$ from (2.24) one deduces the following system of equations:

$$\dot{V}_1 + i(m-\mathscr{D})V_1 - \mathscr{G}V_3 = 0, \tag{2.25a}$$

$$\dot{V}_2 + i(m-\mathscr{D})V_2 - \mathscr{G}V_4 = 0, \tag{2.25b}$$

$$\dot{V}_3 - i(m-\mathscr{D})V_3 + \mathscr{G}V_1 = 0, \tag{2.25c}$$

$$\dot{V}_4 - i(m-\mathscr{D})V_4 + \mathscr{G}V_2 = 0. \tag{2.25d}$$

From (2.12a) we also write the equations for the invariants S, P and $A=\bar{\psi}\bar{\gamma}^5\bar{\gamma}^0\psi$

$$\dot{S}_0 - 2\mathscr{G}A_0 = 0, \tag{2.26a}$$

$$\dot{P}_0 - 2(m-\mathscr{D})A_0 = 0, \tag{2.26b}$$

$$\dot{A}_0 + 2(m-\mathscr{D})P_0 + 2\mathscr{G}S_0 = 0, \tag{2.26c}$$

where $S_0=\tau S,\ P_0=\tau P,$ and $A_0=\tau A.$ The Eq. (2.26) leads to the following relation

$$S^2+P^2+A^2 = C^2/\tau^2, \qquad C^2 = \text{const}. \tag{2.27}$$

Giving the concrete form of F from (2.25) one writes the components of the spinor functions explicitly and using the solutions obtained one can write the components of the spinor current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{2.28}$$

The component j^0

$$j^0 = \frac{1}{\tau}[V_1^*V_1 + V_2^*V_2 + V_3^*V_3 + V_4^*V_4], \tag{2.29}$$

defines the charge density of the spinor field that has the following chronometric-invariant form

$$\rho = (j_0 \cdot j^0)^{1/2}. \tag{2.30}$$

The total charge of the spinor field is defined as

$$Q = \int_{-\infty}^{\infty} \rho\sqrt{-^3 g}\, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \rho\tau\mathscr{V}, \tag{2.31}$$

where $\mathscr{V}$ is the volume. From the spin tensor

$$S^{\mu\nu,\varepsilon} = \frac{1}{4}\bar{\psi}\{\gamma^{\varepsilon}\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^{\varepsilon}\}\psi \tag{2.32}$$

one finds the chronometric invariant spin tensor

$$S_{\rm ch}^{ij,0} = (S_{ij,0}S^{ij,0})^{1/2}, \tag{2.33}$$

and the projection of the spin vector on k axis

$$S_k = \int\limits_{-\infty}^{\infty} S_{\rm ch}^{ij,0}\sqrt{-{}^3 g}\,{\rm d}x{\rm d}y{\rm d}z = S_{\rm ch}^{ij,0}\tau V. \tag{2.34}$$

Let us now solve the Einstein equations. To do it we first write the expressions for the components of the energy-momentum tensor explicitly:

$$T_0^0 = mS - \lambda F(I,J) + \varepsilon, \tag{2.35a}$$

$$T_1^1 = \mathscr{D}S + \mathscr{G}P - \lambda F(I,J) - p' + 2\eta\frac{\dot{a}}{a}, \tag{2.35b}$$

$$T_2^2 = \mathscr{D}S + \mathscr{G}P - \lambda F(I,J) - p' + 2\eta\frac{\dot{b}}{b}, \tag{2.35c}$$

$$T_3^3 = \mathscr{D}S + \mathscr{G}P - \lambda F(I,J) - p' + 2\eta\frac{\dot{c}}{c}. \tag{2.35d}$$

In account of (2.35) subtracting (2.14a) from (2.14b), one finds the following relation between a and b

$$\frac{a}{b} = D_1 \exp\left(X_1\int\frac{{\rm e}^{-2\kappa\int\eta{\rm d}t}}{\tau}{\rm d}t\right). \tag{2.36}$$

Analogically, one finds

$$\frac{b}{c} = D_2 \exp\left(X_2\int\frac{{\rm e}^{-2\kappa\int\eta{\rm d}t}}{\tau}{\rm d}t\right), \quad \frac{c}{a} = D_3 \exp\left(X_3\int\frac{{\rm e}^{-2\kappa\int\eta{\rm d}t}}{\tau}{\rm d}t\right). \tag{2.37}$$

Here D_1, D_2, D_3, X_1, X_2, X_3, are integration constants, obeying

$$D_1D_2D_3 = 1, \quad X_1 + X_2 + X_3 = 0. \tag{2.38}$$

In view of (2.38) from (2.36) and (2.37) we write the metric functions explicitly [21]

$$a(t) = (D_1/D_3)^{1/3}\tau^{1/3}\exp\left[\frac{X_1 - X_3}{3}\int\frac{{\rm e}^{-2\kappa\int\eta{\rm d}t}}{\tau(t)}{\rm d}t\right], \tag{2.39a}$$

$$b(t) = (D_1^2 D_3)^{-1/3} \tau^{1/3} \exp\left[-\frac{2X_1 + X_3}{3} \int \frac{e^{-2\kappa\int \eta dt}}{\tau(t)} dt\right], \tag{2.39b}$$

$$c(t) = (D_1 D_3^2)^{1/3} \tau^{1/3} \exp\left[\frac{X_1 + 2X_3}{3} \int \frac{e^{-2\kappa\int \eta dt}}{\tau(t)} dt\right]. \tag{2.39c}$$

As one sees from (2.39a), (2.39b) and (2.39c), for $\tau = t^n$ with $n > 1$ the exponent tends to unity at large t, and the anisotropic model becomes an isotropic one.

Further we will investigate the existence of the singularity (singular point) of the gravitational case, which can be done by investigating the invariant characteristics of the space-time. In general relativity these invariants are composed from the curvature tensor and the metric one. In a 4D Riemann space-time there are 14 independent invariants. Instead of analyzing all 14 invariants, one can confine this study only in 3, namely the scalar curvature $I_1 = R$, $I_2 = R^R_{\mu\nu}\mu\nu$, and the Kretschmann scalar $I_3 = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$. At any regular space-time point, these three invariants I_1, I_2, I_3 should be finite. Let us rewrite these invariants in detail.

For the Bianchi *I* metric one finds the scalar curvature

$$I_1 = R = -2\left(\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c} + \frac{\dot a}{a}\frac{\dot b}{b} + \frac{\dot b}{b}\frac{\dot c}{c} + \frac{\dot c}{c}\frac{\dot a}{a}\right). \tag{2.40}$$

Since the Ricci tensor for the BI metric is diagonal, the invariant $I_2 =$ $= R_{\mu\nu} R^{\mu\nu} \equiv R^{\nu}_{\mu} R^{\mu}_{\nu}$ is a sum of squares of diagonal components of Ricci tensor, *i.e.*,

$$I_2 = [(R_0^0)^2 + (R_1^1)^2 + (R_2^2)^2 + (R_3^3)^2], \tag{2.41}$$

with the components of the Ricci tensor being given by (2.9).

Analogically, for the Kretschmann scalar in this case we have $I_3 = R^{\mu\nu}_{\alpha\beta} R^{\alpha\beta}_{\mu\nu}$, a sum of squared components of all nontrivial $R^{\mu\nu}_{\mu\nu}$, which in view of (2.11) can be written as

$$\begin{aligned} I_3 &= 4\left[\left(R^{01}_{01}\right)^2 + \left(R^{02}_{02}\right)^2 + \left(R^{03}_{03}\right)^2 + \left(R^{12}_{12}\right)^2 + \left(R^{23}_{23}\right)^2 + \left(R^{31}_{31}\right)^2\right] \\ &= 4\left[\left(\frac{\ddot a}{a}\right)^2 + \left(\frac{\ddot b}{b}\right)^2 + \left(\frac{\ddot c}{c}\right)^2 + \left(\frac{\dot a}{a}\frac{\dot b}{b}\right)^2 + \left(\frac{\dot b}{b}\frac{\dot c}{c}\right)^2 + \left(\frac{\dot c}{c}\frac{\dot a}{a}\right)^2\right]. \end{aligned} \tag{2.42}$$

Let us now express the foregoing invariants in terms of τ. From Eqs. (2.39) we have

$$a_i = A_i \tau^{1/3} \exp\left((Y_i/3)\int \frac{e^{-2\kappa\int \eta dt}}{\tau(t)} dt\right), \tag{2.43a}$$

$$\frac{\dot{a}_i}{a_i} = \frac{\dot{\tau} + Y_i e^{-2\kappa\int \eta dt}}{3\tau} \quad (i = 1, 2, 3), \tag{2.43b}$$

$$\frac{\ddot{a}_i}{a_i} = \frac{3\tau\ddot{\tau} - 2\dot{\tau}^2 - \dot{\tau} Y_i e^{-2\kappa\int \eta dt} - 6\kappa\eta\tau Y_i e^{-2\kappa\int \eta dt} + Y_i^2 e^{-4\kappa\int \eta dt}}{9\tau^2}, \tag{2.43c}$$

i.e., the metric functions a, b, c and their derivatives are in functional dependence with τ. From Eqs. (2.43) one can easily verify that

$$I_1 \propto \frac{1}{\tau^2}, \quad I_2 \propto \frac{1}{\tau^4}, \quad I_3 \propto \frac{1}{\tau^4}.$$

Thus we see that at any space-time point, where $\tau = 0$ the invariants I_1, I_2, I_3 as well as the scalar and spinor fields become infinite, hence the space-time becomes singular at this point.

In what follows, we write the equation for τ and study it in details.

Summation of Einstein equations (2.14a), (2.14b), (2.14c) and (2.14d) multiplied by 3 gives

$$\ddot{\tau} - \frac{3}{2}\kappa\xi\dot{\tau} = \frac{3}{2}\kappa\big(mS + \mathcal{D}S + \mathcal{G}P - 2\lambda F(I, J) + \varepsilon - p\big)\tau - 3\Lambda\tau. \tag{2.44}$$

For the right-hand-side of (2.44) to be a function of τ only, the solution to this equation is well-known [31].

Let us demand the energy-momentum to be conserved, *i.e.*,

$$T^{\nu}_{\mu;\nu} = T^{\nu}_{\mu,\nu} + \Gamma^{\nu}_{\rho\nu} T^{\rho}_{\mu} - \Gamma^{\rho}_{\mu\nu} T^{\nu}_{\rho} = 0, \tag{2.45}$$

which in our case has the form

$$\frac{1}{\tau}(\tau T^0_0)^{\cdot} - \frac{\dot{a}}{a} T^1_1 - \frac{\dot{b}}{b} T^2_2 - \frac{\dot{c}}{c} T^3_3 = 0. \tag{2.46}$$

In account of

$$(m - \mathcal{D})\dot{S}_0 - \mathcal{G}\dot{P}_0 = 0$$

which follows from (2.26), after a little manipulation from (2.46) we obtain

$$\dot{\varepsilon} + \frac{\dot{\tau}}{\tau}\omega - \Big(\xi + \frac{4}{3}\eta\Big)\frac{\dot{\tau}^2}{\tau^2} + 4\eta(\kappa T^0_0 - \Lambda) = 0, \tag{2.47}$$

where

$$\omega = \varepsilon + p, \tag{2.48}$$

is the thermal function.

Let us now in analogy with the Hubble constant introduce a quantity H, such that

$$\frac{\dot{\tau}}{\tau} = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = 3H. \tag{2.49}$$

Then (2.44) and (2.47) in account of (2.35) can be rewritten as

$$\dot{H} = \frac{\kappa}{2}(3\xi H - \omega) - (3H^2 - \kappa\varepsilon) + \frac{\kappa}{2}(mS + \mathscr{D}S + \mathscr{G}P - 2\lambda F(I,J)) - \Lambda, \tag{2.50a}$$

$$\dot{\varepsilon} = 3H(3\xi H - \omega) + 4\eta(3H^2 - \kappa\varepsilon) - 4\eta[\kappa(mS - \lambda F(I,J)) - \Lambda]. \tag{2.50b}$$

Thus, the metric functions are found explicitly in terms of τ and viscosity. To write τ and the components of the spinor field as well and the scalar one we have to specify F in $\mathscr{L}_{\rm int}$. In the next section we explicitly solve Eqs. (2.25) and (2.50) for some concrete value of F.

The Eqs. (2.50) can be written in terms of dynamical scalar as well. For this purpose let us introduce the dynamical scalars such as the expansion and the shear scalar as usual

$$\theta = u^{\mu}_{;\mu}, \quad \sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \tag{2.51}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2}(u_{\mu;\alpha}P^{\alpha}_{\nu} + u_{\nu;\alpha}P^{\alpha}_{\mu}) - \frac{1}{3}\theta P_{\mu\nu}. \tag{2.52}$$

Here P is the projection operator obeying

$$P^2 = P. \tag{2.53}$$

For the space-time with signature (+, –, –, –) it has the form

$$P_{\mu\nu} = g_{\mu\nu} - u_{\mu}u_{\nu}, \quad P^{\mu}_{\nu} = \delta^{\mu}_{\nu} - u^{\mu}u_{\nu}. \tag{2.54}$$

For the BI metric the dynamic scalar has the form

$$\theta = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = \frac{\dot{\tau}}{\tau}, \tag{2.55}$$

and

$$2\sigma^2 = \frac{\dot{a}^2}{a^2} + \frac{\dot{b}^2}{b^2} + \frac{\dot{c}^2}{c^2} - \frac{1}{3}\theta^2. \tag{2.56}$$

In account of (2.39) one can also rewrite share scalar as

$$2\sigma^2 \frac{6(X_1^2 + X_1X_3 + X_3^2)}{9\tau^2} e^{-4\kappa\int\eta dt}. \tag{2.57}$$

From (2.14d) now yields

$$\frac{1}{3}\theta^2 - \sigma^2 = \kappa[mS - \lambda F(I,J) + \varepsilon] - \Lambda \tag{2.58}$$

The Eqs. (2.50) now can be written in terms of θ and σ as follows

$$\dot{\theta} = \frac{3\kappa}{2}(\xi\theta - \omega) - \frac{3\kappa}{2}(mS - \mathcal{D}S - \mathcal{G}P + 2\lambda F(I,J)) - 3\sigma^2, \tag{2.59a}$$

$$\dot{\varepsilon} = \theta(\xi\theta - \omega) + 4\eta\sigma^2. \tag{2.59b}$$

Note that the Eqs. (2.59) without spinor and scalar field contributions coincide with the ones given in [12].

III. SOME SPECIAL SOLUTIONS

In this section we first solve the spinor field equations for some special choice of F, which will be given in terms of τ. Thereafter, we will study the system (2.50) in details and give explicit solution for some special cases.

A. SOLUTIONS TO THE SPINOR FIELD EQUATIONS

As one sees, introduction of viscous fluid has no direct effect on the system of spinor field equations (2.25). Viscous fluid has an implicit influence on the system through τ. A detailed analysis of the system in question can be found in [21]. Here we just write the final results.

1. Case with $F = F(I)$

Here we consider the case when the nonlinear spinor field is given by $F = F(I)$. As in the case with minimal coupling from (2.26a) one finds

$$S = \frac{C_0}{\tau}, \quad C_0 = \text{const.} \tag{3.1}$$

For the components of spinor field we find [21]

$$\begin{aligned} \psi_1(t) &= \frac{C_1}{\sqrt{\tau}} e^{-i\beta}, \quad \psi_2(t) = \frac{C_2}{\sqrt{\tau}} e^{-i\beta}, \\ \psi_3(t) &= \frac{C_3}{\sqrt{\tau}} e^{i\beta}, \quad \psi_4(t) = \frac{C_4}{\sqrt{\tau}} e^{i\beta}, \end{aligned} \tag{3.2}$$

with C_i being the integration constants and are related to C_0 as $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$. Here $\beta = \int (m - \mathcal{D})\mathrm{d}t$.

For the components of the spin current from (2.28) we find

$$j^0 = \frac{1}{\tau}[C_1^2 + C_2^2 + C_3^2 + C_4^2], \quad j^1 = \frac{2}{a\tau}[C_1C_4 + C_2C_3]\cos(2\beta),$$
$$j^2 = \frac{2}{b\tau}[C_1C_4 - C_2C_3]\sin(2\beta), \quad j^3 = \frac{2}{c\tau}[C_1C_3 - C_2C_4]\cos(2\beta),$$

whereas, for the projection of spin vectors on the X, Y and Z axis we find

$$S^{23,0} = \frac{C_1C_2 + C_3C_4}{bc\tau}, \quad S^{31,0} = 0, \quad S^{12,0} = \frac{C_1^2 - C_2^2 + C_3^2 - C_4^2}{2ab\tau}.$$

Total charge of the system in a volume $\mathcal{V}$ in this case is

$$Q = [C_1^2 + C_2^2 + C_3^2 + C_4^2]\mathcal{V}. \tag{3.3}$$

Thus, for $\tau \neq 0$ the components of spin current and the projection of spin vectors are singularity-free and the total charge of the system in a finite volume is always finite. Note that, setting $\lambda = 0$, *i.e.*, $\beta = mt$ in the foregoing expressions one get the results for the linear spinor field.

2. Case with *F* = *F*(*J*)

Here we consider the case with $F = F(J)$. In this case we assume the spinor field to be massless. Note that, in the unified nonlinear spinor theory of Heisenberg, the massive term remains absent, and according to Heisenberg, the particle mass should be obtained as a result of the quantization of the spinor prematter [34]. In the nonlinear generalization of the classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines the total energy (or mass) of the nonlinear field system. Thus without losing the generality we can consider the massless spinor field putting $m = 0$. Then from (2.26b) one gets

$$P = D_0/\tau, \quad D_0 = \text{const}. \tag{3.4}$$

In this case the spinor field components take the form

$$\begin{aligned} \psi_1 &= \frac{1}{\sqrt{\tau}}(D_1 e^{i\sigma} + iD_3 e^{-i\sigma}), \quad \psi_2 = \frac{1}{\sqrt{\tau}}(D_2 e^{i\sigma} + iD_4 e^{-i\sigma}), \\ \psi_3 &= \frac{1}{\sqrt{\tau}}(iD_1 e^{i\sigma} + D_3 e^{-i\sigma}), \quad \psi_4 = \frac{1}{\sqrt{\tau}}(iD_2 e^{i\sigma} + D_4 e^{-i\sigma}). \end{aligned} \tag{3.5}$$

The integration constants D_i are connected to D_0 by $D_0 = 2(D_1^2 + D_2^2 - D_3^2 - D_4^2)$. Here we set $\sigma = \int \mathcal{G}\, dt$.

For the components of the spin current from (2.28) we find

$$j^0 = \frac{2}{\tau}[D_1^2 + D_2^2 + D_3^2 + D_4^2], \quad j^1 = \frac{4}{a\tau}[D_2D_3 + D_1D_4]\cos(2\sigma),$$

$$j^2 = \frac{4}{b\tau}[D_2 D_3 - D_1 D_4]\sin(2\sigma), \quad j^3 = \frac{4}{c\tau}[D_1 D_3 - D_2 D_4]\cos(2\sigma),$$

whereas, for the projection of spin vectors on the *X*, *Y* and *Z* axis we find

$$S^{23,0} = \frac{2(D_1 D_2 + D_3 D_4)}{bc\tau}, \quad S^{31,0} = 0, \quad S^{12,0} = \frac{D_1^2 - D_2^2 + D_3^2 - D_4^2}{2ab\tau}$$

We see that for any nontrivial τ as in the previous case the components of the spin current and the projection of the spin vectors are singularity-free and the total charge of the system in a finite volume is always finite.

B. DETERMINATION OF τ

In this subsection we simultaneously solve the system of equations for τ and ε. For simplicity we set $\kappa = 1$. Since setting $m = 0$ in the equations for $F = F(I)$ one comes to the case when $F = F(J)$, we consider the case with F being the function of I only. Let F be the power function of S, *i.e.*, $F = S^n$. As was established earlier, in this case $S = C_0/\tau$, or setting $C_0 = 1$ simply $S = 1/\tau$. Evaluating $\mathscr{D}$ in terms of τ we then come to the following system of equations

$$\ddot{\tau} = \frac{3}{2}\xi\dot{\tau} + \frac{3}{2}\left(\frac{m}{\tau} + \frac{\lambda(n-2)}{\tau^{n-1}} + \varepsilon - p\right)\tau - 3\Lambda\tau, \tag{3.6a}$$

$$\dot{\varepsilon} = -\frac{\dot{\tau}}{\tau}\omega + \left(\xi + \frac{4}{3}\eta\right)\frac{\dot{\tau}^2}{\tau^2} - 4\eta\left[\frac{m}{\tau} - \frac{\lambda}{\tau^n} - \Lambda\right], \tag{3.6b}$$

or in terms of H

$$\dot{\tau} = 3H\tau, \tag{3.7a}$$

$$\dot{H} = \frac{1}{2}(3\xi H - \omega) - (3H^2 - \varepsilon) + \frac{1}{2}\left(\frac{m}{\tau} + \frac{\lambda(n-2)}{\tau^{n-1}}\right) - \Lambda, \tag{3.7b}$$

$$\dot{\varepsilon} = 3H(3\xi H - \omega) + 4\eta(3H^2 - \varepsilon) - 4\eta\left[\frac{m}{\tau} - \frac{\lambda}{\tau^n} - \Lambda\right]. \tag{3.7c}$$

Here η and ξ are the bulk and shear viscosity, respectively and they are both positively definite, *i.e.*,

$$\eta > 0, \quad \xi > 0. \tag{3.8}$$

They may be either constant or function of time or energy. We consider the case when

$$\eta = A\varepsilon^{\alpha}, \quad \xi = B\varepsilon^{\beta}, \tag{3.9}$$

with A and B being some positive quantities. For p we set as in perfect fluid,

$$p = \zeta\varepsilon, \quad \zeta \in (0, 1]. \tag{3.10}$$

Note that in this case $\zeta \neq 0$, since for dust pressure, hence temperature is zero, that results in vanishing viscosity.

In account of (2.48), (3.9) and (3.10) the Eqs. (3.6) and (3.7) now can be rewritten as

$$\ddot{\tau} = \frac{3}{2}B\varepsilon^{\beta}\dot{\tau} + \frac{3}{2}\left(\frac{m}{\tau} + \frac{\lambda(n-2)}{\tau^{n-1}} + (1-\zeta)\varepsilon\right)\tau - 3\Lambda\tau, \tag{3.11a}$$

$$\dot{\varepsilon} = -(1+\zeta)\varepsilon\frac{\dot{\tau}}{\tau} + \left(B\varepsilon^{\beta} + \frac{4}{3}A\varepsilon^{\alpha}\right)\frac{\dot{\tau}^2}{\tau^2} - 4A\varepsilon^{\alpha}\left[\frac{m}{\tau} - \frac{\lambda}{\tau^n} - \Lambda\right], \tag{3.11b}$$

or in terms of H

$$\dot{\tau} = 3H\tau, \tag{3.12a}$$

$$\dot{H} = \frac{1}{2}(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon) - (3H^2 - \varepsilon) + \frac{1}{2}\left(\frac{m}{\tau} + \frac{\lambda(n-2)}{\tau^{n-1}}\right) - \Lambda, \tag{3.12b}$$

$$\dot{\varepsilon} = 3H(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon) + 4A\varepsilon^{\alpha}(3H^2 - \varepsilon) - 4\eta\left[\frac{m}{\tau} - \frac{\lambda}{\tau^n} - \Lambda\right]. \tag{3.12c}$$

The system (3.7) without spinor field have been extensively studied in the literature either partially [9,12,13] or as a whole [11]. Here we try to solve the system (3.6) for some particular choice of parameters.

1. Case with shear viscosity

Let us first consider the case when $\eta = 0$. We also demand $\dot{\tau}/\tau$ to be small enough. Then overlooking $\dot{\tau}^2/\tau^2$ from (3.6b) in view of (3.10) one finds

$$\varepsilon = \frac{\varepsilon_0}{\tau^{1+\zeta}}. \tag{3.13}$$

In view of (3.9) and (3.13), Eq. (3.6a) now can be written as

$$\ddot{\tau} = f_1(\tau)\dot{\tau} + f_0(\tau), \tag{3.14}$$

where we set

$$f_1(\tau) = \frac{3B}{2}\frac{\varepsilon_0^{\beta}}{\tau^{\beta(1+\zeta)}},$$

and

$$f_0(\tau) = \frac{3}{2}\left(m + \frac{\lambda(n-2)}{\tau^{(n-2)}} + \frac{(1-\zeta)\varepsilon_0}{\tau^{\zeta}}\right) - 3\Lambda\tau.$$

Let us introduce a new function μ such that

$$\dot{\tau} = \mu(\tau). \tag{3.15}$$

Eqn. (3.14) can then be written as

$$\mu\mu'_{\tau} = f_1(\tau)\mu + f_0(\tau), \tag{3.16}$$

Further setting $\mu(\tau) = \nu(\tau) + \overline{f}_1$ where $\overline{f}_1 = \int f_1(\tau)d\tau$ we find

$$(\nu + \overline{f}_1)\nu'_{\tau} = f_0. \tag{3.17}$$

Finally, for $f_0 \neq 0$ assuming that $\nu(\tau) = \upsilon(\varsigma)$ where $\varsigma = \int f_0 d\tau$ we finally obtain

$$(\upsilon + \overline{f}_1)\upsilon'_{\varsigma} = 1. \tag{3.18}$$

For some special cases Eqn. (3.18) allows solution in quadrature [31].

IV. CONCLUSION

We consider the self consistent system of nonlinear spinor and gravitational fields within the framework of Bianchi type-I cosmological model filled with viscous fluid. The spinor filed nonlinearity is taken to be some power law of the invariants of bilinear spinor forms, namely $I = S^2 = (\overline{\psi}\psi)^2$ and $J = P^2 = (i\overline{\psi}\gamma^5\psi)^2$. Solutions to the corresponding equations are given in terms of the volume scale of the BI space-time, *i.e.*, in terms of $\tau = abc$. The system of equations for determining τ, energy-density of the viscous fluid ε and Hubble parameter H has been worked out. Exact solution to the aforementioned system has been given only for the case of share viscosity. As one sees from (2.50) or (2.59), the system in question is a multi-parametric one and may have several solutions depending on the choice of the problem parameters. In the near future we plan to study this system thoroughly both from analytical and numerical point of views.

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