Scalar Field with Induced Nonlinearity: Regular Solutions and their Stability*

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Exact particle-like static, spherically and/or cylindrically symmetric solutions to the equations of interacting scalar and electromagnetic field system have been obtained in external gravitational field. In particular, we considered Freedman-Robertson-Walker (FRW) space-time as an external homogenous and isotropic gravitational field whereas the homogeneous and anisotropic Universe is given by the Gödel model. The solutions obtained have been thoroughly studied for different types of interaction term. It has been shown that in FRW space-time equations with different interaction terms may have stable solutions while within the scope of Gödel model only the droplet-like configurations may be stable, if they are located in the region where $g^{00} > 0$.

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1. INTRODUCTION

The concept of soliton as regular localized stable solutions of nonlinear differential equations is being widely utilized in pure science [1]. One of the fields to apply the soliton concept is the elementary particle physics, where the soliton solutions of nonlinear field equations are used as the simplest models of extended particles [2, 3]. Development of general relativity (GR) and quantum field theory (QFT) leads to the increasing interest to study the role of gravitational field in elementary particle physics. To obtain and study the properties of regular localized solutions to the nonlinear classical field equations is motivated mainly by a hope to create a consistent, divergence-free theory. These solutions, as was remarked by Rajaraman [4] give us one of the ways of modeling elementary particles as extended objects with complicated spatial structure. In such attempts it is natural to treat the field nonlinearity not only as a tool for avoiding the theoretical difficulties (such as singularities) but also as a reflection of real properties of physical system. It should be also emphasized that the complete description of elementary particles with all their physical characteristics (e.g., magnetic momentum) can be given only in the framework of interacting field theory [5]. The effect of gravitational fields on the properties of regular localized solutions significantly depends on the symmetry of the system. In this paper we present some regular particle-like

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solutions for an interacting system of scalar and electromagnetic fields, confining ourselves to static, spherically and/or cylindrically symmetric configurations.

2. FUNDAMENTAL EQUATIONS

So that the field equations possess regular solutions it is necessary to introduce nonlinear terms, describing the field interactions, in the Lagrangian. We consider the nonlinear generalization of the theory that is related to the introduction of direct interaction between neutral scalar and electromagnetic fields. The decay process like $\pi^0 \to 2\gamma$, described by the effective Lagrangian [6]

$$L_{\rm int} = \varphi_{\pi^0} F_{\alpha\beta} F^{*\alpha\beta},$$

indicates to the possibility of such generalization. Thus we consider a system with the Lagrangian

$$L = \frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha} - \frac{1}{16\pi} F_{\alpha\beta}F^{\alpha\beta}\Psi(\varphi) \tag{2.1}$$

where $\Psi(\varphi)$ is some arbitrary function characterizing interaction between the scalar (φ) and electromagnetic ($F_{\mu\nu}$) fields takes the form

$$\Psi(\varphi) = 1 + \lambda\Phi(\varphi).$$

As is seen, for $\lambda = 0$, $\Psi(\varphi) \equiv 1$ and we have the system with minimal coupling. Note that the Lagrangian (2.1) describes the system of fields with positive definite energy if $\Psi(\varphi) \geq 0$. This kind of interaction has been thoroughly discussed in [7].

Let us write the scalar and the electromagnetic field equations corresponding to the Lagrangian(2.1)

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\Bigl(\sqrt{-g}g^{\nu\mu}\frac{\partial\varphi}{\partial x^\mu}\Bigr) + \frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\Psi_\varphi = 0, \quad \Psi_\varphi = \frac{\partial\Psi}{\partial\varphi} \tag{2.2}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\Bigl(\sqrt{-g}F^{\nu\mu}\Psi(\varphi)\Bigr) = 0 \tag{2.3}$$

The corresponding energy-momentum tensor reads

$$T_\mu^\nu = \varphi_{,\mu}\varphi^{,\nu} - \frac{1}{4\pi}F_{\mu\beta}F^{\nu\beta}\Psi(\varphi) - \delta_\mu^\nu\Bigl[\frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha} - \frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\Psi(\varphi)\Bigr] \tag{2.4}$$

3. SPHERICALLY SYMMETRIC CONFIGURATIONS

A. Solutions in FRW Universe

As an external homogenous and isotropic gravitational field we choose the FRW space-time. This Universe is very important as the corresponding cosmological models coincides with observation. The interval in the FRW Universe in general takes the form [8, 9]

$$ds^2 = dt^2 - R^2(t)\Bigl[\frac{dr^2}{1-kr^2} + r^2\bigl\{d\vartheta^2 + \sin^2\vartheta d\phi^2\bigr\}\Bigr] \tag{3.1}$$

Here $R(t)$ defines the size of the Universe, and k takes the values 0 and ± 1. We consider the simple most case putting $R(t) = R =$ constant, which corresponds to the static FRW Universe. In static case $k = 0$ corresponds to usual Minkowski space, $k = +1$ describes the close Einstein Universe [10] and $k = -1$ corresponds to the space-time with hyperbolic spatial cross-section. Note that the velocity of light c has been taken to be unity.

As was mentioned earlier, we seek the static, spherically symmetric solutions to the equations (2.2) and (2.3). To this end we assume that the scalar field is the function of r only, i.e. $\varphi = \varphi(r)$ and the electromagnetic field possesses only one component $F_{10} = \partial A_0/\partial r = A'$.

Under the assumption made above, the solution to the equation (2.3) reads

$$F^{01} = \bar{q} P(\varphi) \frac{\sqrt{1 - kr^2}}{R^3 r^2} \tag{3.2}$$

where $\bar{q}$ is the constant of integration and $P(\varphi) = 1/\Psi(\varphi)$. Putting (3.2) into (2.2) for the scalar field we obtain the equation with "induced nonlinearity" [11, 12]

$$(1 - kr^2)\varphi'' + \frac{2 - 3kr^2}{r}\varphi' - \frac{2q^2}{R^2 r^4} P_\varphi = 0, \quad q^2 = \frac{\bar{q}^2}{16\pi} \tag{3.3}$$

This equation can be written in the form

$$\frac{\partial^2 \varphi}{\partial z^2} - \frac{2q^2}{R^2} P_\varphi = 0 \tag{3.4}$$

with the first integral

$$\frac{\partial \varphi}{\partial z} = \frac{2q}{R}\sqrt{P + C_0} \tag{3.5}$$

where we substitute $z = \sqrt{1/r^2 - k}$. Here C_0 is the constant of integration, which under the regularity condition of T_0^0 at the center turns to be trivial, i.e., $C_0 = 0$. Finally we write the solution to the scalar field equation in quadrature

$$\int \frac{\partial \varphi}{\sqrt{P}} = \frac{2q}{R}(z - z_0) \tag{3.6}$$

In accordance with (3.2) and (3.5) from (2.4) we find the density of field energy of the system

$$T_0^0 = \frac{4q^2 P}{R^4 r^4} \tag{3.7}$$

and total energy of the material field system

$$E_f = \int T_0^0 \sqrt{-^3 g}\, d^3\mathbf{x} = -8\pi q \int \sqrt{P} d\varphi \tag{3.8}$$

Thus, we see that the energy density T_0^0 and total energy E_f of the configurations obtained do not depend on the conventional values of the parameter k. As one sees, to write the scalar (φ) and vector (A) functions as well as the energy density (T_0^0) and energy of the material fields (E_f) explicitly, one has to give $P(\varphi)$ in explicit form. Here we will give a detailed analysis for some concrete forms of $P(\varphi)$. Let us choose $P(\varphi)$ in the form

$$P(\varphi) = P_0 \cos^2\left(\frac{\lambda\varphi}{2}\right) \tag{3.9}$$

with λ being the interaction parameter. Inserting (3.9) into (3.4) we get the sin-Gordon type equation [13]

$$\frac{\partial^2 \varphi}{\partial z^2} + \frac{\lambda q^2 P_0}{4R^2} \sin(\lambda\varphi) = 0 \tag{3.10}$$

with the solution

$$\varphi(z) = \frac{2}{\lambda} \arcsin \tanh[b(z+z_1)], \quad b = \frac{\lambda q \sqrt{P_0}}{R}, \quad z_1 = \mathrm{const} \tag{3.11}$$

Let us analyze the solution (3.11). It can be shown that $\lim\limits_{r\to 0} \varphi = \pi/\lambda$ for all k. As one sees, the solution possesses meaning only in the region where $z = \sqrt{1/r^2 - k} > 0$. It means, in case of $k = +1$ the configuration confines in the interval $0 \le r \le 1$. Then the asymptotic behavior of the solution (3.11) can be written as

$$\varphi \to \begin{cases} 0, \; k=-1, & z_1 = -1, \quad r \to \infty \\ 0, \quad k=0, & z_1 = 0, \quad r \to \infty \\ 0, \quad k=+1, & z_1 = 0, \quad r \ge 1 \end{cases} \tag{3.12}$$

From (3.8) we find the total energy of the system $E_f = -16\pi q\sqrt{P_0}/\lambda$. For the choice of $P(\varphi)$ in the form

$$P = \lambda(a^2 - \varphi^2)^2 \tag{3.13}$$

with λ being the coupling constant and a being some arbitrary constant, from (3.4) we obtain

$$4\lambda a^2 \varphi - 4\lambda\varphi^3 + \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{3.14}$$

The equation (3.14) can be seen as an MKdV one. Indeed, a KdV equation

$$\frac{\partial u}{\partial t} + \alpha u^p \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0, \tag{3.15}$$

can always be converted to

$$-Du + \alpha \frac{u^{p+1}}{p+1} + \beta \frac{d^2 u}{dz^2} = 0 \tag{3.16}$$

if one looks for stationary solution of the form $u = u(z)$ where $z = x - Dt$. In our particular case $p = 2$ and the equation (3.14) is an MKdV one. The scalar field function in this case has the form

$$\varphi(z) = a\tanh[\sqrt{\lambda} ab(z+z_2)], \quad b = \frac{2q}{R} \tag{3.17}$$

Taking into account that $z = \sqrt{1/r^2 - k}$ one sees that at the origin $\lim\limits_{r\to 0} \varphi = a$ whereas at the asymptotic region for different value of k we get

$$\varphi \to \begin{cases} 0, \; k=-1, & z_2 = -1, \quad r \to \infty \\ 0, \quad k=0, & z_2 = 0, \quad r \to \infty \\ 0, \quad k=+1, & z_2 = 0, \quad r \ge 1 \end{cases} \tag{3.18}$$

From (3.8) we find the total energy of the system to be $E_f = -16\pi q\sqrt{\lambda} a^3/3$. A specific type of solution to the nonlinear field equations in flat space-time was obtained in a series of interesting articles [14]. These solutions are known as droplet-like solutions or simply droplets. The distinguishing property of these solutions is the availability of some sharp boundary defining the space domain in which the material field happens to be located, i.e., the field is zero beyond this area. It was found that the solutions mentioned exist in field theory with specific interactions that can be considered as an effective one, generated by initial interactions of unknown origin. In contrast to the widely known soliton-like solutions, with field functions and energy density asymptotically tending to zero at spatial infinity, the solutions in question vanish at a finite distance from the center of the system (in the case of spherical symmetry) or from the axis (in the case of cylindrical symmetry). Thus, there exists a sphere or cylinder with critical radius r_0 outside of which the fields disappear. Therefore the field configurations have a droplet-like structure [11, 14, 15].

To obtain the droplet-like configuration we choose a very specific type of interaction function $P(\varphi)$ which has the form [16]

$$P(\varphi) = J^{2-4/\sigma}\Bigl(1 - J^{2/\sigma}\Bigr)^2 \tag{3.19}$$

where $J = \lambda\varphi, \quad \sigma = 2n+1, \quad n = 1,\,2\cdots$. Putting (3.19) into (3.6) for φ one gets

$$\varphi(z) = \frac{1}{\lambda}\Bigl[1 - \exp\Bigl(-\frac{4q\lambda}{R\sigma}(z - z_0)\Bigr)\Bigr]^{\sigma/2} \tag{3.20}$$

Recalling that $z = \sqrt{1/r^2 - k}$ from (1) we see that at $r \to 0$ the scalar field φ takes the value $\varphi(0) \to 1/\lambda$ and at $r \to r_c = 1/\sqrt{z_0^2 + k}$, the scalar field function becomes trivial, i.e., $\varphi(r_c) \to 0$. It is obvious that for $r > r_c$ the value of the square bracket turns out to be negative and $\varphi(r)$ becomes imaginary, since σ is an odd number. Since we are interested in real φ only, without loss of generality we may assume the value of φ to be zero for $r \geq r_c$, the matching at $r = r_c$ (i.e., $z = z_0$) being smooth. Note that, for $k = +1$, the scalar field is confined in the region $0 \leq r \leq 1$, as it was in previous two cases. The total energy of the droplet we obtain from (3.8) has the form

$$E_f = \frac{4\pi q}{\lambda(\sigma - 1)} \tag{3.21}$$

As is seen from (3.21), the value of the total energy does not depend on the size of the droplet, it means droplets of different linear size share the same total energy.

B. Stability problem

To study the stability of the configurations obtained we write the linearized equations for the radial perturbations of scalar field assuming that

$$\varphi(r,t) = \varphi(r) + \xi(r,t), \quad \xi \ll \varphi \tag{3.22}$$

Putting (3.22) into (2.2) in view of (3.3) we get the equation for $\xi(r,t)$

$$\ddot{\xi} + 3\frac{\dot R}{R}\dot{\xi} - \frac{1 - kr^2}{R^2}\xi'' - \frac{2 - 3kr^2}{rR^2}\xi' + \frac{q^2 P_{\varphi\varphi}}{R^4 r^4}\xi = 0 \tag{3.23}$$

The second term in (3.23) is zero since we assume the FRW space-time to be static one putting $R =$ constant. Assuming that

$$\xi(r,t) \approx v(r)\exp(-i\Omega t), \quad \Omega = \omega/R \tag{3.24}$$

from (3.23) we obtain

$$(1-kr^2)v'' - \frac{2-3kr^2}{r}v' + \Big[\omega^2 - \frac{q^2 P_{\varphi\varphi}}{R^2 r^4}\Big]v = 0 \tag{3.25}$$

The substitution

$$\eta(\zeta) = r\cdot v(r), \quad \zeta = \frac{1}{\sqrt{k}}\arcsin(\sqrt{k}r) \tag{3.26}$$

leads the equation (3.25) to the Liouville one [17]

$$\eta_{\zeta\zeta} + \big(\omega^2 - V(\varphi)\big)\eta = 0, \quad V(\varphi) = -k + \frac{q^2 P_{\varphi\varphi}}{R^2\zeta^4}\Big(\frac{\sqrt{k}\zeta}{\sin(\sqrt{k}\zeta)}\Big)^4 \tag{3.27}$$

For the interaction term (3.9) we see that

$$V(\varphi) > 0, \quad \text{if and only if} \quad \tanh^2 b(z+z_1) > \frac{k\zeta^4 R^2}{P_0 q^2\lambda^2}\Big(\frac{\sin(\sqrt{k}\zeta)}{\sqrt{k}\zeta}\Big)^4 + \frac{1}{2} \tag{3.28}$$

Thus we find that the equations with trigonometric nonlinearity contain stable solutions. Given $P(\varphi)$ in the form (3.13) we find that

$$V(\varphi) > 0, \quad \text{if} \quad \tanh^2 b(z+z_2) > \frac{k\zeta^4 R^2}{12 q^2\lambda}\Big(\frac{\sin(\sqrt{k}\zeta)}{\sqrt{k}\zeta}\Big)^4 + \frac{a^2}{3} \tag{3.29}$$

As is seen from (3.29) the equations with polynomial type of nonlinearity too contain some stable solutions. For the droplet-like configurations, i.e., for the interacting term $P(\varphi)$ given by (3.19), it can be shown that the potential

$$\lim_{r\to 0} V(\varphi) \to +\infty, \quad \lim_{r\to r_c} V(\varphi) \to +\infty \tag{3.30}$$

beginning with $\sigma \geq 5$. It means that the droplet-like configurations (1) with $\sigma \geq 5$ are stable for the class of perturbation, vanishing at $r = 0$ and $r = r_c$.

4. CYLINDRICALLY SYMMETRIC CONFIGURATIONS

A. Solutions in Gödel Universe

In the previous section we studied the possibility of formation of regular localized configuration in homogenous and isotropic FRW Universe. Let us now continue our study in the homogenous but anisotropic Universe. In particular we consider the model proposed by Gödel. The linear element of Gödel Universe in cylindrical coordinates reads [18]

$$ds^2 = dt^2 - d\rho^2 + \frac{1}{\Omega^2}[\sinh^4\Omega\rho - \sinh^2\Omega\rho]d\phi^2 - \frac{\sqrt{8}}{\Omega}\sinh^2\Omega\rho d\phi dt - dz^2 \tag{4.1}$$

where the constant Ω is related with the angular velocity ω: $\omega = \sqrt{2}\Omega$. This form of linear element of the four-dimensional homogenous space S directly exhibits its rotational symmetry, since the $g_{\mu\nu}$ do not depend on ϕ. It is easy to find

$$\lim_{\Omega\to 0}\sqrt{-g} = \lim_{\Omega\to 0}\frac{1}{2\Omega}\sinh(2\Omega\rho) \to \rho$$

i.e. at $\omega \to 0$ Gödel Universe transfers to the flat one.

As in spherically symmetric case, here too we seek the static solutions to the equations (2.2) and (2.3) assuming the scalar field to be the function of ρ only, i.e., $\varphi = \varphi(\rho)$ and the electromagnetic possesses only one component $F_{10} = \partial A_0/\partial\rho = A'$. For the electromagnetic field in this case we find

$$F^{01} = 2\Omega DP/\sinh(2\Omega\rho), \quad D = \text{const} \tag{4.2}$$

The scalar field equation (2.2) with regards to (4.2) reads

$$\frac{\partial^2\varphi}{\partial\rho^2} + 2\Omega\coth(2\Omega\rho)\frac{\partial\varphi}{\partial\rho} = \frac{8q^2\Omega^2 P_\varphi}{\sinh^2(2\Omega\rho)}, \quad q^2 = D^2/16\pi \tag{4.3}$$

Putting $y = \frac{1}{2\Omega}\ln\tanh(\Omega\rho)$ from (4.3) one gets

$$\frac{\partial^2\varphi}{\partial y^2} - 8q^2\Omega^2 P_\varphi = 0 \tag{4.4}$$

with the first integral

$$\frac{\partial\varphi}{\partial y} = \pm 4q\Omega\sqrt{P + D_0} \tag{4.5}$$

Here D_0 is the constant of integration, which under the regularity condition of T_0^0 at the center turns to be trivial, i.e., $D_0 = 0$. Finally we write the solution to the scalar field equation in quadrature

$$\int\frac{\partial\varphi}{\sqrt{P}} = 4q\Omega(y - y_0) \tag{4.6}$$

In accordance with (4.2) and (4.5) from (2.4) we find the density of field energy and the total energy of the system

$$T_0^0 = \frac{16q^2\Omega^2 P}{\sinh^2(2\Omega\rho)} \tag{4.7}$$

$$E_f = 8\pi q\int\sqrt{P}d\varphi \tag{4.8}$$

As in the previous case, to write the scalar (φ) and vector (A) functions as well as the energy density (T_0^0) and energy of the material fields (E_f) explicitly, one has to give $P(\varphi)$ in explicit form. Here again we will thoroughly study the solutions obtained for different concrete forms of $P(\varphi)$. Choosing $P(\varphi)$ in the form (3.9), i.e.,

$$P(\varphi) = P_0\cos^2\left(\frac{\lambda\varphi}{2}\right) \tag{4.9}$$

with λ being the interaction parameter, from (4.4) we get the sin-Gordon type equation

$$\frac{\partial^2 \varphi}{\partial y^2} + 4\lambda q^2 \Omega^2 P_0 \sin(\lambda\varphi) = 0 \tag{4.10}$$

The solution to this equation can be written in the form

$$\varphi(\rho) = \frac{4}{\lambda}\Bigl[\arctan\Bigl(\frac{\tanh \Omega\rho}{\Omega\rho_0}\Bigr)^{\alpha} - \frac{\pi}{4}\Bigr], \quad \alpha = \lambda q \sqrt{P_0} \tag{4.11}$$

where ρ_0 is the constant of integration, giving the size of the system. Without losing the generality we can choose $\alpha > 0$. Then one finds $\lim\limits_{\rho\to 0}\varphi = -\pi/\lambda$. For $\rho > 0$ the field φ steadily increases up to π/λ. In particular, at spatial infinity we get $\lim\limits_{\rho\to\infty}\varphi = 0$. In this case $P(\varphi_\infty) = 1$ which corresponds to the exclusion of interaction at spatial infinity. The total energy of the system in this case coincides with that of in FRW Universe, i.e., $E_f = -16\pi q\sqrt{P_0}/\lambda$. Choosing $P(\varphi)$ in the form (3.13), i.e.,

$$P = \lambda(a^2 - \varphi^2)^2 \tag{4.12}$$

from (4.4) we as in FRW case again obtain MKdV type equation with the solution

$$\varphi(\rho) = a\frac{\tanh^{\alpha}(\Omega\rho) - 1}{\tanh^{\alpha}(\Omega\rho) + 1}, \quad \alpha = 4aq\sqrt{\lambda} \tag{4.13}$$

From (4.13) it is clear that $\lim\limits_{\rho\to 0}\varphi \to -a$ and $\lim\limits_{\rho\to\infty}\varphi \to 0$ From (4.8) we find the total energy of the system to be $E_f = -16\pi q\sqrt{\lambda}a^3/3$ as it was in FRW case. The choice of the interaction term $P(\varphi)$ in the form (3.19), i.e.,

$$P(\varphi) = J^{2-4/\sigma}\bigl(1 - J^{2/\sigma}\bigr)^2 \tag{4.14}$$

where $J = \lambda\varphi, \quad \sigma = 2n+1, \quad n = 1, 2\cdots$, leads to the following expression for scalar field

$$\varphi(\rho) = \frac{1}{\lambda}\Bigl[1 - \Bigl(\frac{\tanh(\Omega\rho)}{\tanh(\Omega\rho_0)}\Bigr)^{\alpha}\Bigr]^{\sigma/2}, \quad \alpha = \pm\frac{4\lambda q}{\sigma} \tag{4.15}$$

where ρ_0 is an arbitrary constant. For $\alpha > 0$ the solution possesses physical meaning at $\rho < \rho_0$ and becomes meaningless at $\rho > \rho_0$. Outside the cylinder $\rho = \rho_0$ one can put $\varphi \equiv 0$. This trivial solution is stitched with the solution at $\rho = \rho_0$ under condition $\varphi'(\rho_0) = 0$, which fulfills if and only if $4\lambda|q| > \sigma$. Consequently, at $\rho > \rho_0$ the Lagrangian becomes physically meaningless, however its limiting value at $\rho \to \rho_0 - 0$ is equal to zero. Continuing it at $\rho > \rho_0$, one can consider the field be totally trivial in this area. Thus we get the droplet-like configuration. The field φ steadily decreases from $\varphi(0) = 1/\lambda$ to $\varphi(\rho_0) = 0$ with $\varphi'(0) = 0$ (for $\sigma \geq 3$) and $\varphi'(\rho_0) = 0$. The total energy of the "droplet" is defined as

$$E_f = \frac{4\pi q}{\lambda(\sigma - 1)} \tag{4.16}$$

which remains unaltered even in flat space-time ($\Omega = 0$). It means that the "droplet" does not feel Gödel gravitational field.

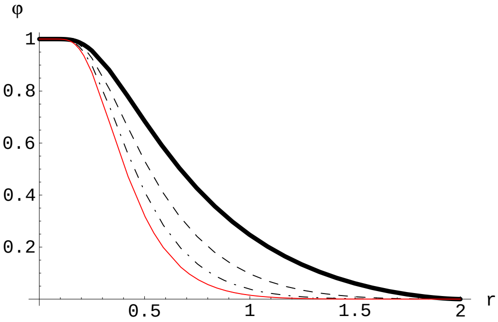

FIG. 1: view of droplet-like configurations for different values of σ. Here the thick-line, dash-line, dash-dot-line and thin-line correspond to the value of $\sigma = 3, 5, 7, 9$ respectively.

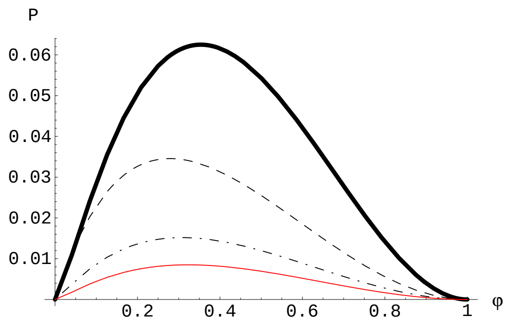

FIG. 2: view of interaction function $P(\varphi)$ providing droplet-like configurations.

B. Stability problem

Let us now study the stability of the configuration obtained. In doing so we consider the perturbed scalar field $\delta\varphi = \chi(\rho, t)$ that leaves the cylindrical-symmetry of the system unbroken. The linearized equation for the perturbed scalar field looks

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\Big(\sqrt{-g}g^{\mu\nu}\chi_{,\nu}\Big) + \frac{q^2}{2(-g)}P_{\varphi\varphi}\chi = 0 \tag{4.17}$$

Since, for the case considered, $g^{00} = (1 - \sinh^2\Omega\rho)/\cosh^2\Omega\rho$, it is clear that the type of equation (4.17) changes on the surface where $\sinh\Omega\rho = 1$. The Cauchy problem for (4.17) is incorrect by Hadamard in the region, where $g^{00} < 0$. In connection with this only the droplet-like solutions can be stable by Lyapunov, if they are located in the region where $g^{00} > 0$ [11]. Assuming that

$$\chi(\rho, t) = v(\rho)\exp(-i\varepsilon t) \tag{4.18}$$

from (4.17) we get

$$v'' + 2\Omega\coth(2\Omega\rho)v' + [\frac{1 - \sinh^2\Omega\rho}{\cosh^2\Omega\rho}\varepsilon^2 - \frac{2\Omega^2 q^2}{\sinh^2(2\Omega\rho)}P_{\varphi\varphi}]v = 0 \tag{4.19}$$

The substitution

$$\eta(\zeta) = \xi(\rho)\cdot v(\rho) \tag{4.20}$$

where

$$\zeta = \int\frac{\sqrt{1 - \sinh^2\Omega\rho}}{\cosh\Omega\rho}d\rho, \quad \xi = \big[4\sinh^2\Omega\rho\big(1 - \sinh^2\Omega\rho\big)\big]^{1/4}$$

leads the equation for perturbed field to the Liouville one

$$\frac{\partial^2\eta}{\partial\zeta^2} + \big[\varepsilon^2 - V\big]\eta = 0 \tag{4.21}$$

Here the effective potential $V(\varphi)$ takes the form

$$V(\varphi) = \frac{\Omega^2}{(1-\sinh^2\Omega\rho)}\Big\{\frac{q^2 P_{\varphi\varphi}}{2\sinh^2\Omega\rho} + \frac{(4\sinh^6\Omega\rho - 16\sinh^4\Omega\rho + 3\sinh^2\Omega\rho - 1)\cosh^2\Omega\rho}{4\sinh^2\Omega\rho\,(1-\sinh^2\Omega\rho)}\Big\}$$

For the interaction function, chosen in the form (3.19), one finds

$$P_{\varphi\varphi} = \lambda^2\big(\frac{2\sigma^2 - 12\sigma + 16}{\sigma^2 J^{4/\sigma}} - \frac{4\sigma^2 - 12\sigma + 8}{\sigma^2 J^{2/\sigma}} + 2\big)$$

Taking into account that $\lim\limits_{\rho\to 0} J \to 1$ and $\lim\limits_{\rho\to\rho_0} J \to 0$, it can be shown that

$$\begin{aligned}\lim_{\rho\to 0} V(\varphi) \to &\quad +\infty \quad \text{for} \quad \sigma < 4q\lambda\\ \lim_{\rho\to\rho_0} V(\varphi) \to &\quad +\infty \quad \text{for} \quad \sigma \ge 5\end{aligned}$$

Here we used the fact that $g^{00} > 0$, i.e., $\sinh\Omega\rho < |1|$. Thus, as in the previous case we find that the droplet like configurations are stable for $5 \le \sigma < 4|q|\lambda$.

5. CONCLUSION

We obtained the regular particle-like solutions to the scalar field equations with induced nonlinearity in external gravitational fields described by Freedman-Robertson-Walker and Gödel Universes respectively. Beside the usual solitons, a special type of regular localized configurations, known as droplets, have been obtained. It has been shown that the droplet-like configurations possess limited energy density and finite total energy and the droplets of different linear sizes up to the soliton share one and the same total energy. It is noteworthy to notice that in the spherically symmetrical case (i.e., in the FRW Universe) at $r_c \to \infty$ for $k = 0$ droplet transfers to usual solitonian solution, while for $k = \pm 1$ this is not the case. It has also been shown that in FRW space-time equations with different type of nonlinearities may contain stable solutions, whereas in case of Gödel Universe only the droplet-like configurations may be stable. It is noteworthy to remark that in FRW space-time with $k = +1$ the field function is confined in the region $0 \le r \le 1$ independent to the choice of interaction function $P(\varphi)$. Further we plan to derive some exotic solutions like anti-droplet and also consider the case with R being the function of time.

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