Electrodynamics with Toroid Polarization*

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A modified system of equations of electrodynamics of continuous media has been obtained. Beside the Lagrangian one an alternative gauge-like formalism has been developed to introduce the toroid moment contributions in the equations obtained. The two potential formalism that was worked out by us earlier has been developed further where along with the two vector potentials we introduce two scalar potentials thus taking into account all the four basic equations of electromagnetism.

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1. INTRODUCTION

Ya. Zel'dovich [1] was the first to introduce *anapole* in connection with the global electromagnetic properties of a toroid coil that are impossible to ascribe within the charge or magnetic dipole moments in spite of explicit axial symmetry of the toroid coil. Further, by 1974 Dubovik and Cheskov [2] finally defined the toroid moment in the framework of classical electrodynamics. As it came out, namely it (toroid moment) corresponds to the point-like toroidal solenoid, whereas anapole contains, additional to the toroid moment, a linear element of direct current centered to it [2, 3]. Toroid polarization is made evident in different condensed matter by a large number of investigations. For magnetic media, we note the recent measurement of toroid moment in $Ga_{2-x}Fe_xO_3$ [4] and Cr_2O_3 [5].

Moreover, a principally new property of media known as *aromagnetism* was observed in a class of organic substances, suspended either in water or in other liquids [6]. Later, it was shown that this phenomena of aromagnetism cannot be explained in a standard way, e.g., by ferromagnetism, since the organic molecules do not possess magnetic moments of either orbital or spin origin. It was also shown that the origin of aromagnetism is the interaction of vortex electric field induced by alternative magnetic one with the axial toroid moments of the fragment C_6 in aromatic elements [7].

These experimental results force to introduce toroid polarizations in the framework of electrodynamics continuous media that in its part inevitably leads to the modification of the basic equations. Recently we introduced toroid moments in Maxwell and Schrödinger equations exploiting Lagrangian formalism [8]. Here we give a brief description of this formalism. Beside this we elaborate an alternative method to introduce toroid moments in the equation of electromagnetism. Further we develop two potential formalism suggested by us earlier [8–10].

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2. INTRODUCTION OF TOROID POLARIZATIONS INTO THE EQUATIONS OF ELECTROMAGNETISM

In this section we give two alternative description of introduction of toroid polarizations into the equations of electromagnetism. The first one is based on the Lagrangian formalism, whereas we call the second method as a gauge-like one.

A. Lagrangian Formalism

As a starting point we consider the interacting system of electromagnetic field and non-relativistic charged particles given by the Lagrangian density [11]

$$\begin{aligned} L &= L_{\rm par} + L_{\rm rad} + L_{\rm int} \qquad (2.1)\\ L_{\rm par} &= \frac{1}{2}\sum_{\alpha} m_\alpha \dot{\mathbf{q}}_\alpha^2 - \frac{1}{2}\sum_{\alpha\neq\beta} \frac{e_\alpha\, e_\beta}{|\mathbf{q}_\alpha - \mathbf{q}_\beta|}\\ L_{\rm rad} &= \frac{1}{8\pi}\int \Big[\frac{\dot{\mathbf{A}}^2}{c^2} - (\mathbf{\nabla}\times\mathbf{A})^2\Big]\, d\mathbf{r}\\ L_{\rm int} &= \frac{1}{c}\int \mathbf{J}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})\, d\mathbf{r} = \sum_{\alpha} \frac{e_\alpha}{c}\, \dot{\mathbf{q}}_\alpha\cdot\mathbf{A}(q_\alpha, t). \end{aligned}$$

Here $L_{\rm par}$ is the Lagrangian appropriate to a system of charged particles interacting solely through instantaneous Coulomb force; it has the simple form of "kinetic energy minus potential energy". $L_{\rm rad}$ is the Lagrangian for an external radiation field far removed from the charges and currents, and has the form of "electric field energy minus magnetic field energy". The interaction Lagrangian $L_{\rm int}$ couples the particle variables to the field ones. It can be easily verified that variation with respect to the particle coordinates gives the second law of Newton with the Lorentz force

$$m_\alpha\, \ddot{\mathbf{q}}_\alpha = e_\alpha\, \mathbf{E}(\mathbf{q}_\alpha, t) + \frac{e_\alpha}{c}\dot{\mathbf{q}}_\alpha \times \mathbf{B}(\mathbf{q}_\alpha, t). \qquad (2.2)$$

Variation of the Lagrangian (2.1) with respect to field variables gives the equation of motion for the vector potential

$$\mathbf{\nabla}\times\mathbf{\nabla}\times\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} \qquad (2.3)$$

Defining $\mathbf{B} = \mathbf{\nabla}\times\mathbf{A}$ and $\mathbf{E} = -\dot{\mathbf{A}}/c$ one obtains

$$\mathbf{\nabla}\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \qquad (2.4)$$

It should be emphasized that $\mathbf{E}$ in (2.2) and (2.4) is the transverse part of the total electric field. The longitudinal electric field in question is entirely electrostatic.

The Hamiltonian, corresponding to the Lagrangian (2.1) reads

$$\begin{aligned} H[\mathbf{\Pi}, \mathbf{A}; p, q] &= \sum_\alpha \mathbf{p}_\alpha \cdot \dot{\mathbf{q}}_\alpha + \int \mathbf{\Pi} \cdot \dot{\mathbf{A}} d\mathbf{r} - L \\ &= \sum_\alpha \frac{1}{2m_\alpha} \big[\mathbf{p}_\alpha - \frac{e_\alpha}{c} \mathbf{A}(\mathbf{q}, t)\big]^2 + \frac{1}{2} \sum_{\alpha \neq \beta} \frac{e_\alpha e_\beta}{|\mathbf{q}_\alpha - \mathbf{q}_\beta|} \\ &+ \frac{1}{8\pi} \int \big[(4\pi c \mathbf{\Pi})^2 + (\mathbf{\nabla} \times \mathbf{A})^2\big] d\mathbf{r}, \end{aligned} \tag{2.5}$$

where the corresponding conjugate momenta are

$$\mathbf{p}_\alpha = m \dot{\mathbf{q}}_\alpha + (e_\alpha / c) \mathbf{A}(\mathbf{q}, t), \quad \mathbf{\Pi}(\mathbf{r}) = (4\pi c^2)^{-1} \dot{\mathbf{A}}.$$

It is well known that in classical dynamics the addition of a total time derivative to a Lagrangian leads to a new Lagrangian with the equations of motion unaltered. Lagrangians obtained in this manner are treated to be equivalent. In general, the Hamiltonians following from the equivalent Lagrangians are different. Even the relationship between the conjugate and the kinetic momenta may be changed [12]. Moreover, let us notice that the basic equations of any new theory cannot be introduced strictly deductively. Usually, either they are postulated in differential form based on the partial integral conservation laws or transformations of basic dynamical variables, whose initial definitions usually have some analog in mechanics. Furthermore, we need to do so not only by inertia of thinking but also because of the fact that most of our measurements have its objects as individual particles or use them as test one. The situation is the same in electromagnetism and in gravitation. In general geometrical interpretation of dynamical variables plays the crucial role. An equivalent Lagrangian to that of (2.1) is [8]

$$L^{\mathrm{equiv}} = L - \frac{1}{c} \frac{d}{dt} \big[\int \{\mathbf{P}(\mathbf{r}) + \mathbf{\nabla} \times \mathbf{T}^e(\mathbf{r})\} \cdot \mathbf{A}(\mathbf{r}) \, dV \big], \tag{2.6}$$

where the toroid contribution has been taken into account. Here $\mathbf{T}^e$ is axial toroid polarization (ATM) is electrical by nature (corresponding to a distribution of toroid dipole moments of electric type). The field conjugate to the vector potential $\mathbf{A}$ is now

$$4\pi c \mathbf{\Pi} = -(4\pi c)^{-1} \Big(\mathbf{E} + 4\pi (\mathbf{P} + \mathbf{\nabla} \times \mathbf{T}^e) \Big)$$

Since only the free field $\mathbf{E}$ is generated due to the change of magnetic field $\mathbf{B}$ one writes

$$\mathbf{\nabla} \times \boldsymbol{\mathcal{D}} = -\frac{1}{c} \dot{\mathbf{B}} + 4\pi (\mathbf{\nabla} \times \mathbf{P} + \mathbf{\nabla} \times \mathbf{\nabla} \times \mathbf{T}^e), \tag{2.7}$$

under $\mathbf{\nabla} \times \mathbf{E} = -\dot{\mathbf{B}}/c$. Here we define

$$\boldsymbol{\mathcal{D}} = \mathbf{D} + 4\pi \mathbf{\nabla} \times \mathbf{T}^e, \quad \boldsymbol{\mathcal{E}} = \mathbf{E} + 4\pi \mathbf{\nabla} \times \mathbf{T}^e, \quad \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} \tag{2.8}$$

The new Lagrangian is a function of the variables $\mathbf{q}_\alpha$, $\dot{\mathbf{q}}_\alpha$ and a functional of the field variables $\mathbf{A}$, $\dot{\mathbf{A}}$, and the equations of motion follow from the variational principle applied to the latter. As a result, the Euler-Lagrange equations of motion has the form [8]

$$\mathbf{\nabla} \times \mathbf{B} = \frac{1}{c} \dot{\boldsymbol{\mathcal{D}}} + \frac{4\pi}{c} \mathbf{j}_{\mathrm{free}} + 4\pi \Big(\mathbf{\nabla} \times \mathbf{M} + \mathbf{\nabla} \times \mathbf{\nabla} \times \mathbf{T}^m) \Big) \tag{2.9}$$

Here the currents were divided into free and bound state (due to electric polarization and magnetization) one as [13]

$$\mathbf{J}(\mathbf{r}) = \mathbf{j}_{\rm free} + c\,\boldsymbol{\nabla}\times\,\mathbf{M}(\mathbf{r}) + \dot{\mathbf{P}}(\mathbf{r}) \tag{2.10}$$

and an additional local condition on $\mathbf{T}^e$ is imposed:

$$\boldsymbol{\nabla}\times\,\mathbf{T}^{m,e} \equiv \pm\frac{1}{c}\dot{\mathbf{T}}^{e,m} \tag{2.11}$$

where $\mathbf{T}^m$ is the toroid dipole polarization vector of magnetic type. The relation (2.11) demands some comments. Both $\mathbf{T}^e$ and $\mathbf{T}^m$ represent the closed isolated lines of electric and magnetic fields. So their dynamics at point may be reexpressed by the usual differential relations similar to the free Maxwell equations for free electric and magnetic fields [14, 15]). However, remark that signs here are opposite to the corresponding one in Maxwell equations because the direction of electric dipole is usually accepted to be chosen opposite to its inner electric field [16].

Defining the auxiliary field $\boldsymbol{\mathcal{H}}$ in the following way

$$\boldsymbol{\mathcal{H}} = \mathbf{H} - 4\pi\boldsymbol{\nabla}\times\,\mathbf{T}^m, \quad \boldsymbol{\mathcal{B}} = \mathbf{B} - 4\pi\boldsymbol{\nabla}\times\,\mathbf{T}^m, \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \tag{2.12}$$

then it deduces to

$$\boldsymbol{\nabla}\times\,\boldsymbol{\mathcal{H}} \;=\; \frac{1}{c}\dot{\boldsymbol{\mathcal{D}}} + \frac{4\pi}{c}\,\mathbf{j}_{\rm free}$$

But the latter formula is unsatisfactory from the physical point of view. It is easy to image the situation when $\mathbf{B}$ and $\mathbf{M}$ are absent, because the medium may be composed from isolated aligned dipoles $\mathbf{T}^m$ [17–19] and each $\mathbf{T}^m$ is the source of free-field (transverse-longitudinal) potential but not $\mathbf{B}$ [20]. So the transition to the description by means of potentials is inevitable.

The Hamiltonian, corresponding to the equivalent Lagrangian in this case reads

$$\begin{aligned} H^{\rm equiv}[\boldsymbol{\Pi},\mathbf{A};p,q] \;=\; & \sum_\alpha \frac{1}{2m_\alpha}\Big[\mathbf{p}_\alpha - \frac{e_\alpha}{c}\mathbf{A}(\mathbf{q},t)\Big]^2 + \frac{1}{2}\sum_{\alpha\neq\beta}\frac{e_\alpha\, e_\beta}{|\mathbf{q}_\alpha - \mathbf{q}_\beta|} \\ & + \frac{1}{8\pi}\int\Big\{[4\pi(\mathbf{P} + \boldsymbol{\nabla}\times\,\mathbf{T}^e) - \mathbf{D}]^2 + (\boldsymbol{\nabla}\times\,\mathbf{A})^2\Big\}d\mathbf{r} \\ & + \frac{1}{c}\int\mathbf{J}\cdot\mathbf{A}d\mathbf{r} - \int\mathbf{M}\cdot\mathbf{B}d\mathbf{r} - \int\mathbf{B}\cdot\boldsymbol{\nabla}\times\,\mathbf{T}^m d\mathbf{r}. \end{aligned} \tag{2.13}$$

B. Gauge-like Transformation

The Maxwell equations for electromagnetic fields in media can be written as

$$\boldsymbol{\nabla}\times\,\mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} \;=\; \frac{4\pi}{c}\mathbf{j}_{\rm free} \tag{2.14}$$

$$\boldsymbol{\nabla}\cdot\,\mathbf{D} \;=\; 4\pi\rho \tag{2.15}$$

$$\boldsymbol{\nabla}\times\,\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \;=\; 0 \tag{2.16}$$

$$\boldsymbol{\nabla}\cdot\,\mathbf{B} \;=\; 0 \tag{2.17}$$

where

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \tag{2.18}$$
$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} \tag{2.19}$$

In the previous subsection we have introduced toroid polarizations into Maxwell equation through Lagrangian formalism. In doing so we first constructed an equivalent Lagrangian. Here we are going to do the same using in an alternative way which looks rather a gauge transformation. To this end we introduce two vectors $\mathbf{T}^m$ and $\mathbf{T}^e$ of toroid dipole polarizations of the vector (magnetic) and axial (electric) types, respectively such that

$$\mathbf{P} \Longrightarrow \mathbf{P} + \boldsymbol{\nabla} \times \mathbf{T}^e \tag{2.20}$$
$$\mathbf{M} \Longrightarrow \mathbf{M} + \boldsymbol{\nabla} \times \mathbf{T}^m \tag{2.21}$$

It can be easily shown that the system (2 B) is invariant under the transformation (2 B) if we impose the additional condition (2.11), i.e.,

$$\boldsymbol{\nabla} \times \mathbf{T}^{e,m} \equiv \pm\frac{1}{c}\frac{\partial \mathbf{T}^{m,e}}{\partial t} \tag{2.22}$$

In account of both (2 B) and (2.22) we obtain the complete system (2 B) as

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{H}} = \frac{1}{c}\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_{\text{free}} \tag{2.23}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{D}} = 4\pi\rho \tag{2.24}$$
$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} = -\frac{1}{c}\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} \tag{2.25}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{B}} = 0 \tag{2.26}$$

As is seen the equations (2.23) and (2.25) of the system (2 B) completely coincide with (2.9) and (2.7) of the previous subsection. Thus we introduced toroid polarizations in Maxwell equations using two different formalisms. However, there are some fine circumstances where the toroid sources generate the free-field potential (Franz's, Ederly-Sidhu's, Bohm-Aharonov's ones) that is also known as *longitudinal-transverse vector potential* (LTVP) [21–23]. In the last form of the equations they are omitted since the equations (2.23) and (2.24) take into account the contributions of $\boldsymbol{\nabla} \times \mathbf{T}^{m,e} \neq 0$ only. Therefore, we need in transition to potential formulation of the basic equations ultimately.

3. TWO POTENTIAL FORMALISM

It is commonly believed that the divergence equations of the Maxwell system are "redundant". Recently Krivsky a.o. [24] claimed that to describe the free electromagnetic field it is sufficient to consider the curl-subsystem of Maxwell equations since the equalities $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ are fulfilled identically. Contrary to this statement, Jiang and Co [25] proved that the divergence equations are not redundant and that neglecting these equations is at the origin of spurious solutions in computational electromagnetics immediately. Here we construct generalized formulation of Maxwell equations including both curl and divergence subsystems. In this section we develop two potential formalism (a similar formalism was developed by us earlier with the curl-subsystem taken into account only). Note that in the ordinary one potential formalism $(\mathbf{A}, \varphi)$ the second set of Maxwell equations are fulfilled identically. So that all the four Maxwell equations bring their contribution individually, in our view, one has to rewrite the Maxwell equation in terms of two vector and two scalar potentials.

Because of introduction of toroid polarizations now the original $\mathbf{B}$ and $\mathbf{D}$ supplemented by $\boldsymbol{\nabla} \times \mathbf{T}^m$ and $\boldsymbol{\nabla} \times \mathbf{T}^e$ respectively, hence should be reinterpreted. It means the deduction of the equation of evolution by inserting $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and $\mathbf{E} = -\dot{\mathbf{A}}/c$ is valid no longer and we have to introduce some new potential that could explain the new $\mathbf{B}$ and $\mathbf{D}$. To this end we introduce so-called double potential [8–10]. As was mentioned, due to introduction of toroid polarizations the vectors $\mathbf{B}$ and $\mathbf{D}$ should be redefined. We denote these new quantities as $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$, respectively. In account of it, the system (2 B) should be rewritten as

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} &= \frac{1}{c}\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} + \frac{4\pi}{c}\big(\mathbf{j}_{\text{free}} + \frac{\partial \mathbf{P}}{\partial t} + c\,\boldsymbol{\nabla} \times \mathbf{M}\big) && (3.1)\\
\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} &= 4\pi(\rho - \boldsymbol{\nabla} \cdot \mathbf{P}) && (3.2)\\
\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} &= -\frac{1}{c}\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} && (3.3)\\
\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{B}} &= 0 && (3.4)
\end{aligned}$$

Before developing the two potential formalism we first rewrite system (2 B) with $\mathbf{H}$ and $\mathbf{D}$ replaced by $\mathbf{B}$ and $\mathbf{E}$, respectively, in terms of vector and scalar potentials $\mathbf{A}, \phi$ such that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, $\mathbf{E} = -\nabla\phi - (1/c)(\partial\mathbf{A}/\partial t)$. Following any text book we can write

$$\begin{aligned}
\Box\,\mathbf{A} &= -\frac{4\pi}{c}\mathbf{j}_{\text{tot}} = -\frac{4\pi}{c}\big[\mathbf{j}_{\text{free}} + \frac{\partial \mathbf{P}}{\partial t} + c\,\boldsymbol{\nabla} \times \mathbf{M}\big] && (3.5)\\
\Box\phi &= -4\pi\big[\rho - \boldsymbol{\nabla} \cdot \mathbf{P}\big] && (3.6)
\end{aligned}$$

under Lorentz gauge, i.e., $\boldsymbol{\nabla} \cdot \mathbf{A} + (1/c)(\partial\phi/\partial t) = 0$. Here $\Box = \nabla^2 - (1/c^2)(\partial^2/\partial t^2)$. Note that to obtain (3) it is sufficient to consider (2.14) and (2.15) with $\mathbf{H}$ and $\mathbf{D}$ replaced by $\mathbf{B}$ and $\mathbf{E}$ respectively, while the two others, i.e., (2.16) and (2.17), are fulfilled identically. Let us now develop two potential formalism. Two potential formalism was first introduced in [9] and further developed in [8, 10]. In both papers we introduce only two vector potentials $\boldsymbol{\alpha}^m$, $\boldsymbol{\alpha}^e$ and use only the curl-subsystem of the Maxwell equations with the additional condition $\boldsymbol{\nabla} \cdot \boldsymbol{\alpha}^{m,e} = 0$. Thus, in our view our previous version of two potential formalism lack of completeness. In the present paper together with the vector potentials $\boldsymbol{\alpha}^m, \boldsymbol{\alpha}^e$ we introduce two scalar potentials φ^m and φ^e such that

$$\begin{aligned}
\boldsymbol{\mathcal{B}} &= \boldsymbol{\nabla} \times \boldsymbol{\alpha}^m + \frac{1}{c}\frac{\partial \boldsymbol{\alpha}^e}{\partial t} + \nabla\varphi^m, && (3.7)\\
\boldsymbol{\mathcal{E}} &= \boldsymbol{\nabla} \times \boldsymbol{\alpha}^e - \frac{1}{c}\frac{\partial \boldsymbol{\alpha}^m}{\partial t} - \nabla\varphi^e && (3.8)
\end{aligned}$$

Inserting (3) into the system of equations (3) we find

$$\begin{aligned}
\Box\,\boldsymbol{\alpha}^m &= -\frac{4\pi}{c}\big(\mathbf{j}_{\text{free}} + \frac{\partial \mathbf{P}}{\partial t} + c\,\boldsymbol{\nabla} \times \mathbf{M}\big) && (3.9)\\
\Box\,\varphi^m &= 0 && (3.10)\\
\Box\,\boldsymbol{\alpha}^e &= 0 && (3.11)\\
\Box\,\varphi^e &= -4\pi\,(\rho - \boldsymbol{\nabla} \cdot \mathbf{P}) && (3.12)
\end{aligned}$$

under $\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}^{m,e}+(1/c)(\partial\varphi^{e,m}/\partial t)=0$. The general solutions to the systems (3) can be written as (see for example [8, 26]):

$$F(\mathbf{r},t)=-\frac{1}{4\pi}\int\limits_{\text{all space}}\left.\frac{f(\mathbf{r}',t')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right|_{t'=t-|\mathbf{r}-\mathbf{r}'|/c} \tag{3.13}$$

where $F(\mathbf{r},t)$ stands for the unknowns $\boldsymbol{\alpha}^e,\varphi^e,\boldsymbol{\alpha}^m,\varphi^m$, whereas f is the right hand side of the equations (3).

Here we would also like to note that in our previous works [8–10] we introduced two-potentials in the following way

$$\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla}\times\boldsymbol{\alpha}^m+\frac{1}{c}\frac{\partial\boldsymbol{\alpha}^e}{\partial t}, \tag{3.14}$$

$$\boldsymbol{\mathcal{D}} = \boldsymbol{\nabla}\times\boldsymbol{\alpha}^e-\frac{1}{c}\frac{\partial\boldsymbol{\alpha}^m}{\partial t} \tag{3.15}$$

As was mentioned earlier the scalar parts $\varphi^{m,e}$ has not been taken into account. Since we are dealing with moving media and want the equations of electromagnetism with toroid polarizations to be Lorentz covariant, we should consider the pair $(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{E}})$ as in one-potential case rather than the pair $(\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{D}})$.

It is necessary to emphasize that the potential descriptions electrotoroidic and magnetotoroidic media are completely separated. The properties of the magnetic and electric potentials $\boldsymbol{\alpha}^m$ and $\boldsymbol{\alpha}^e$ under the temporal and spatial inversions are opposite [14]. The potential $\boldsymbol{\alpha}^e$ ($\boldsymbol{\alpha}^m$) is related to the toroidness of the medium $\mathbf{T}^e$ ($\mathbf{T}^m$) as $\mathbf{B}$ ($\mathbf{D}$) to $\mathbf{M}$ ($\mathbf{P}$).

Note that if $\boldsymbol{\nabla}\cdot\boldsymbol{\delta}\neq 0$ and there does exist free current in the medium we have to use the direct method for finding all constrains in the theory suggested by Dirac. Dirac applied his method to electrodynamics and found that electromagnetic potentials have only two degrees of freedom described by transverse components of vector potential. This method was developed by Dubovik and Shabanov [23], where classical and quantum dynamics of a system of non-relativistic charged particles were considered.

4. CONCLUSION

The modified equations of electrodynamics containing the contribution toroid polarizations have been obtained. The two-potential formalism for these equations has been developed further. Note that introduction of free magnetic current $\mathbf{j}^m_{\text{free}}$ and magnetic charge ρ^m in the equations (3.3) and (3.4) respectively leads to the equations obtained by Singleton [27, 28].

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