Some Regular Solutions to the Scalar Field Equation with Induced Nonlinearity*

Bijan Saha

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, Dubna
141980 Dubna, Moscow region, Russia†
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Exact particle-like solutions to the interacting system of scalar and electromagnetic field equations within the scope of external Freedman-Robertson-Walker (FRW) space-time have been obtained. In particular, static, spherically symmetric droplet-like configurations have been found and their linearized stability has been proved.

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1. INTRODUCTION

Development of general relativity (GR) and quantum field theory (QFT) leads to the increasing interest to study the role of gravitational field in elementary particle physics. To obtain and study the properties of regular localized solutions to the nonlinear classical field equations is motivated mainly by a hope to create a consistent, divergence-free theory. These solutions, as was remarked by Rajaraman [1] give us one of the ways of modeling elementary particles as extended objects with complicated spatial structure. In such attempts it is natural to treat the field nonlinearity not only as a tool for avoiding the theoretical difficulties (such as singularities) but also as a reflection of real properties of physical system. It should be also emphasized that the complete description of elementary particles with all their physical characteristics (e.g., magnetic momentum) can be given only in the framework of interacting field theory [2].

In this paper we present some regular particle-like solutions within the scope of general relativity for an interacting system of scalar and electromagnetic fields, confining ourselves to static, spherically symmetric configurations.

2. FUNDAMENTAL EQUATIONS AND THEIR SOLUTIONS

As an external homogenous and isotropic gravitational field we choose the FRW space-time. This Universe is very important as the corresponding cosmological models coincides

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†Electronic address: `saha@thsun1.jinr.ru`; URL: `http://thsun1.jinr.ru/~saha/`

with observation. The interval in the FRW Universe in general takes the form [3, 4]

$$ds^2 = dt^2 - R^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2\big\{d\vartheta^2 + \sin^2\vartheta d\phi^2\big\}\Big] \tag{2.1}$$

Here $R(t)$ defines the size of the Universe, and k takes the values 0 and ±1. We consider the simple most case putting $R(t) = R =$ constant, which corresponds to the static FRW Universe. In static case $k = 0$ corresponds to usual Minkowski space, $k = +1$ describes the close Einstein Universe [5] and $k = -1$ corresponds to the space-time with hyperbolic spatial cross-section. Note that the velocity of light c has been taken to be unity.

We consider a system with the Lagrangian

$$L = \frac{R}{2\kappa} + \frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha} - \frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\Psi(\varphi) \tag{2.2}$$

where the first term describes the Einstein gravitational field (R is the scalar curvature, $\kappa = 8\pi G$ is the gravitational constant) and $\Psi(\varphi)$ is some arbitrary function characterizing interaction between the scalar (φ) and electromagnetic ($F_{\mu\nu}$) fields. This kind of interaction has been thoroughly discussed in [6]. The nongravitational part of the Lagrangian(2.2) describes a system with a positive-definite energy if and only if $\Psi \geq 0$.

The scalar and electromagnetic field equations and the energy-momentum tensor corresponding to the Lagrangian(2.2)

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g^{\nu\mu}\frac{\partial\varphi}{\partial x^\mu}\right) + \frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\Psi_\varphi = 0, \quad \Psi_\varphi = \frac{\partial\Psi}{\partial\varphi} \tag{2.3}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}F^{\nu\mu}\right) = 0 \tag{2.4}$$

$$T_\mu^\nu = \varphi_{,\mu}\varphi^{,\nu} - \frac{1}{4\pi}F_{\mu\beta}F^{\nu\beta}\Psi(\varphi) - \delta_\mu^\nu\Big[\frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha} - \frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\Psi(\varphi)\Big] \tag{2.5}$$

As was mentioned earlier, we seek the static, spherically symmetric solutions to the equations (2.3) and (2.4). To this end we assume that the scalar is the function of r only, i.e. $\varphi = \varphi(r)$ and the electromagnetic possesses only one component $F_{10} = \partial A_0/\partial r = A'$.

Under the assumption made above, the solution to the equation (2.4) reads

$$F^{01} = \bar{q}P(\varphi)\frac{\sqrt{1-kr^2}}{R^3r^2} \tag{2.6}$$

where $\bar{q}$ is the constant of integration and $P(\varphi) = 1/\Psi(\varphi)$.

Putting (2.6) into (2.3) for the scalar field we obtain the equation with "induced nonlinearity" [7]

$$(1-kr^2)\varphi'' + \frac{2-3kr^2}{r}\varphi' - \frac{2q^2}{R^2r^4}P_\varphi = 0 \tag{2.7}$$

Substituting $z = \sqrt{1/r^2 - k}$ we rewrite this equation as

$$\frac{\partial^2\varphi}{\partial z^2} - \frac{2q^2}{R^2}P_\varphi = 0 \tag{2.8}$$

The first integral of (2.8) gives

$$\frac{\partial\varphi}{\partial z} = \frac{2q}{R}\sqrt{P + C_0} \tag{2.9}$$

Here C_0 is the constant of integration, which under the regularity condition of T_0^0 at the center turns to be trivial, i.e., $C_0 = 0$. Finally we write the solution to the scalar field equation in quadrature

$$\int \frac{\partial\varphi}{\sqrt{P}} = \frac{2q}{R}(z - z_0) \tag{2.10}$$

Thus we obtain the general solutions for the scalar and electromagnetic fields.

In accord with (2.6) and (2.9) from (2.5) we find the energy density and the total energy of the field system

$$T_0^0 = \frac{4q^2 P}{R^4 r^4} \tag{2.11}$$

$$E_f = \int T_0^0 \sqrt{-^3 g} d^3\mathbf{x} = -8\pi q \int \sqrt{P} d\varphi \tag{2.12}$$

As, one sees that the energy density T_0^0 and total energy E_f of the configurations obtained do not depend on the conventional values of the parameter k.

Thus the equations for the scalar and electromagnetic fields are completely integrated. As one sees, to write the scalar (φ) and vector (A) functions as well as the energy density (T_0^0) and energy of the material fields (E_f) explicitly, one has to give $P(\varphi)$ in explicit form. Here we will give a detailed analysis for some concrete forms of $P(\varphi)$.

Choosing $P(\varphi)$ in the form

$$P(\varphi) = P_0 \cos^2\bigl(\frac{\lambda\varphi}{2}\bigr) \tag{2.13}$$

with λ being the interaction parameter, from (2.8) we get the sin-Gordon type equation

$$\frac{\partial^2\varphi}{\partial z^2} + \frac{\lambda q^2 P_0}{4R^2}\sin(\lambda\varphi) = 0 \tag{2.14}$$

The solution to this equation takes the form [8]

$$\varphi(z) = \frac{2}{\lambda}\arcsin \mathrm{th}[b(z + z_1)], \quad b = \frac{\lambda q\sqrt{P_0}}{R}, \quad z_1 = \mathrm{const} \tag{2.15}$$

Note that for $z_1 = 0$ we get $\varphi = 0$ and $P = 1$ at spatial infinity.

A specific type of solution to the nonlinear field equations in flat space-time was obtained in a series of interesting articles [9]. These solutions are known as droplet-like solutions or simply droplets. The distinguishing property of these solutions is the availability of some sharp boundary defining the space domain in which the material field happens to be located, i.e., the field is zero beyond this area. It was found that the solutions mentioned exist in field theory with specific interactions that can be considered as effective, generated by initial interactions of unknown origin. In contrast to the widely known soliton-like solutions, with field functions and energy density asymptotically tending to zero at spatial infinity, the solutions in question vanish at a finite distance from the center of the system (in the case of spherical symmetry) or from the axis (in the case of cylindrical symmetry). Thus, there exists a sphere or cylinder with critical radius r_0 outside of which the fields disappear. Therefore the field configurations have a droplet-like structure [9, 10].

Choosing $P(\varphi)$ in the form

$$P(\varphi) = J^{2-4/\sigma}\Big(1 - J^{2/\sigma}\Big)^2, \tag{2.16}$$

where $J = \lambda\varphi, \quad \sigma = 2n+1, \quad n = 1,\, 2\cdots$, for φ one gets:

$$\varphi(z) = \frac{1}{\lambda}\Big[1 - \exp\Big(-\frac{4q\lambda}{R\sigma}(z - z_0)\Big)\Big]^{\sigma/2} \tag{2.17}$$

from which it is obvious that at $r \to 0$, i.e., at $z \to \infty$ the scalar field $\varphi \to 1/\lambda$ and at $r \to r_c$ which corresponds to $z \to z_0$, the scalar field function becomes trivial, i.e., $\varphi(r_c) \to 0$.

3. STABILITY PROBLEM

To study the stability of the configurations obtained we write the linearized equations for the radial perturbations of scalar field assuming that

$$\varphi(r,t) = \varphi(r) + \xi(r,t), \quad \xi \ll \varphi \tag{3.1}$$

Putting (3.1) into (2.3) in view of (2.7) we get the equation for $\xi(r,t)$

$$\ddot{\xi} + 3\frac{\dot{R}}{R}\dot{\xi} - \frac{1-kr^2}{R^2}\xi'' - \frac{2-3kr^2}{rR^2}\xi' + \frac{q^2 P_{\varphi\varphi}}{R^4 r^4}\xi = 0 \tag{3.2}$$

The second term in (3.2) is zero since we assume the FRW space-time to be static one putting $R =$ constant. Assuming that

$$\xi(r,t) \approx v(r)\mathrm{exp}(-i\Omega t), \quad \Omega = \omega/R \tag{3.3}$$

from (3.2) we obtain

$$(1-kr^2)v'' - \frac{2-3kr^2}{r}v' + \Big[\omega^2 - \frac{q^2 P_{\varphi\varphi}}{R^2 r^4}\Big]v = 0 \tag{3.4}$$

The substitution

$$\eta(\zeta) = r\cdot v(r), \quad \zeta = \frac{1}{\sqrt{k}}\arcsin(\sqrt{k}r) \tag{3.5}$$

leads the equation (3.4) to the Liouville one [11]

$$\eta_{\zeta\zeta} + \big(\omega^2 - V(\varphi)\big)\eta = 0, \quad V(\varphi) = -k + \frac{q^2 k^2 P_{\varphi\varphi}}{R^2}\mathrm{cosec}^4(\sqrt{k}\zeta) \tag{3.6}$$

Let us analyze the cases with different $P(\varphi)$.

For $P(\varphi)$ chosen as (2.13) we get

$$P_{\varphi\varphi} = \frac{\lambda^2 P_0}{2}\big[2\mathrm{th}^2[b(z+z_1)] - 1\big] \tag{3.7}$$

We find that for trigonometric nonlinearity the sign of potential $V(\varphi)$ is not uniquely defined, therefore some of the configurations can be unstable. Nevertheless, for some cases, the constant z_1 could be chosen such that $V(\varphi) \geq 0$. Thus for (3.7) we get, $P_{\varphi\varphi} \geq 0$ for $\mathrm{ch} b z_1 \geq \sqrt{2}$. Hence we can conclude that equations with the interacting term type (2.13) contains stable solutions [8].

Let us consider the droplet-like configurations. It can be shown that for the interacting term $P(\varphi)$ given by (2.16), the potential

$$V(\varphi) = -k + \frac{q^2 k^2 P_{\varphi\varphi}}{R^2} \mathrm{cosec}^4(\sqrt{k}\zeta) \to +\infty, \quad \text{as} \quad r \to 0, \quad \text{or} \quad r \to r_c \tag{3.8}$$

beginning with $\sigma \geq 5$. It means that the droplet-like configurations (2.17) with $\sigma \geq 5$ are stable for the class of perturbation, vanishing at $r = 0$ and $r = r_c$.

4. CONCLUSION

Regular, static, particle-like solutions to the scalar field equation with induced nonlinearity have been obtained. It is shown that the energy density and the total energy of the system do not depend on the conventional value of k. In case of trigonometrical nonlinearity we found the soliton-like solutions that can be stable for some special choice of constant. In particular, a special type of regular solution, known as droplet, has been found. Contrary to the usual soliton, droplet possesses sharp boundary. It should be underlined that droplets of different linear sizes up to the soliton with $r_c \to \infty$ share one and the same total energy. It is also noteworthy to notice that at $r_c \to \infty$ for $k = 0$ droplet transfers to usual solitonian solution, while in case of $k = \pm 1$ this type of transition remains absent. It should also be emphasized that introduction of gravitational field enforce the stability of the configurations obtained.

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