PACS 04.20.Jb

DROPLET-LIKE SOLUTIONS TO THE EQUATIONS OF SCALAR NONLINEAR ELECTRODYNAMICS IN GENERAL RELATIVITY[1]

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Summary

It is shown, for the self-consistent system of scalar, electro-magnetic and gravitational fields in general relativity, that the equations of motion admit a special kind of solutions with spherical or cylindrical symmetry. For these solutions, the physical fields vanish and the space-time is flat outside of the critical sphere or cylinder. Therefore, the mass and the electric charge of these configurations are zero.

1 Introduction

We study the existence of regular static solutions, possessing spherical and/or cylindrical symmetry and sharp boundary, to the equations of scalar nonlinear electrodynamics in general relativity. Excluding the scalar field we obtain the effective Lagrangian of nonlinear electrodynamics. Contrary to the widely known soliton-like solutions, with field functions and energy density asymptotically tending to zero at spatial infinity, the solutions in question vanish at a finite distance from the center of the system (in the case of spherical symmetry) or from the axis (in the case of cylindrical symmetry). Thus, there exists the sphere or cylinder with critical radius r_0, outside of which the fields disappear. Therefore the field configurations have the droplet-like structure [1,2]. As is known, there do not exist regular static spherically or cylindrically symmetric configurations within the framework of gauge invariant nonlinear electrodynamics [3]. That is why we consider the generalized model, with the Lagrangian explicitly containing 4-potential $\mathcal{A}_\mu$ $\mu = 0,\ 1,\ 2,\ 3$, thus breaking the gauge invariance inside the critical sphere or cylinder. The corresponding terms appear due to the interaction between the electromagnetic and scalar fields. This interaction being negligible at large distances, the Maxwellian structure of the electromagnetic Lagrangian (and therefore the gauge invariance) is reinstated for $r \gg r_0$.

[1] "Communications in Theoretical Physics", **3** (1), pp. 67-80 (1994).

2 Fundamental Equations

We choose the Lagrangian in the form

$$\mathcal{L} \,=\, \mathcal{R}/2\,\kappa \,-\, (1/16\,\pi)\,[\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} \,+\, 2\,\varphi_{,\alpha}\,\varphi^{,\alpha}\,\Psi(I)\,], \tag{2.1}$$

where $\kappa \,=\, 8\,\pi\,G$ is the Einstein's gravitational constant and the function $\Psi(I)$ of the invariant $I \,=\, \mathcal{A}_\mu\,\mathcal{A}^\mu$ characterizes the interaction between the scalar φ and electromagnetic $\mathcal{A}_\mu$ fields. In the sequel there will not appear any restrictions of the function $\Psi(I)$, thus the Lagrangian (2.1) defines the class of models parameterized by $\Psi(I)$.

The field equations corresponding to the Lagrangian (2.1) read

$$\mathcal{G}_\mu^\nu \,=\, -\,\kappa\,T_\mu^\nu, \tag{2.2}$$

$$\partial_\alpha\,(\sqrt{-g}\,g^{\alpha\beta}\,\varphi_{,\beta}\,\Psi) \,=\, 0, \tag{2.3}$$

$$(1/\sqrt{-g})\,\partial_\beta(\sqrt{-g}\,\mathcal{F}^{\alpha\beta}) \,+\, (\varphi_{,\beta}\,\varphi^{,\beta})\,\Psi_I\,\mathcal{A}^\alpha \,=\, 0, \tag{2.4}$$

where $\Psi_I \,=\, d\Psi/dI$ and $\mathcal{G}_\mu^\nu \,=\, \mathcal{R}_\mu^\nu \,-\, \delta_\mu^\nu\,\mathcal{R}/2$ is the Einstein tensor. One can write the energy-momentum tensor of the interacting matter fields in the form:

$$T_\mu^\nu \,=\, (1/4\pi)\,[\varphi_{,\mu}\,\varphi^{,\nu}\,\Psi(I) \,-\, \mathcal{F}_{\mu\alpha}\,\mathcal{F}^{\nu\alpha} \,+\, \varphi_{,\alpha}\,\varphi^{,\alpha}\,\Psi_I\,\mathcal{A}_\mu\,\mathcal{A}^\mu] +$$

$$+\,\delta_\mu^\nu[(1/16\,\pi)\,(\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} \,+\, 2\,\varphi_{,\alpha}\,\varphi^{,\alpha}\,\Psi(I)\,)]. \tag{2.5}$$

3 Configurations with spherical symmetry

Searching for the static spherically-symmetric solutions to the system of equations (2.2) - (2.4) , we consider the metric in the form:

$$ds^2 \,=\, e^{2\gamma(\xi)}\,dt^2 \,-\, e^{2\alpha(\xi)}\,d\xi^2 \,-\, e^{2\beta(\xi)}[d\theta^2 \,+\, \sin^2\theta\; d\phi^2]. \tag{3.1}$$

It is convenient to choose the radial coordinate ξ in (3.1) to satisfy the subsidiary condition:

$$\alpha \,=\, 2\,\beta \,+\, \gamma. \tag{3.2}$$

With the constraint (3.2) the system of Einstein equations (2.2) reads:

$$e^{-2\alpha}\,(2\beta'' \,-\, U) \,-\, e^{-2\beta} \,=\, -\kappa\,T_0^0, \tag{3.3}$$

$$e^{-2\alpha}\,U \,-\, e^{-2\beta} \,=\, -\kappa\,T_1^1, \tag{3.4}$$

$$e^{-2\alpha}\,(\beta'' \,+\, \gamma'' \,-\, U) \,=\, -\kappa\,T_2^2 \,=\, -\kappa\,T_3^3, \tag{3.5}$$

where $U \,=\, \beta'^2 \,+\, 2\,\beta'\,\gamma'$, and $' \equiv d/d\xi$.

Note that the field functions, as well as the components of the metric tensor depend on the single spatial variable ξ. Assuming the electromagnetic field to be determined by the time component $\mathcal{A}_0 \,=\, \mathcal{A}(\xi)$ of the 4-potential one finds the unique non-trivial component of the field tensor $\mathcal{F}_{10} \,=\, \mathcal{A}'$, and the invariant I reduces to $I \,=\, e^{-2\gamma}\,\mathcal{A}(\xi)$.

One can write the non-zero components of the energy-momentum tensor (2.5) as follows:

$$T_0^0 = (1/8\pi)\, e^{-2\alpha}\, [\mathcal{A}'^2\, e^{-2\gamma} + \varphi'^2(\Psi - 2\,\mathcal{A}^2\, e^{-2\gamma}\, \Psi_I\,)], \tag{3.6}$$

$$T_1^1 = -\,T_2^2 = -\,T_3^3 = (1/8\pi)\, e^{-2\alpha}\, [\mathcal{A}'^2\, e^{-2\gamma} + \varphi'^2\, \Psi]. \tag{3.7}$$

Adding together the equations (3.4) and (3.5) and using the property $T_1^1 + T_2^2 = 0$, one obtains the differential equation

$$\beta'' + \gamma'' - e^{2(\beta+\gamma)} = 0,$$

with the solution

$$e^{-(\beta+\gamma)} = \mathcal{S}(k,\,\xi) = \begin{cases} k^{-1}\,\mathrm{sh}\, k\xi, & k > 0, \\ \xi, & k = 0, \\ k^{-1}\sin k\xi, & k < 0, \end{cases} \tag{3.8}$$

depending on the constant k. Notice that another constant of integration is trivial, so that $\xi = 0$ corresponds to the spatial infinity, where $e^{\gamma} = 1$ and $e^{\beta} = \infty$. Without loss of generality one can choose $\xi > 0$.

The scalar field equation (2.3) has the evident solution

$$\varphi' = C\, P(I), \tag{3.9}$$

where $P(I) = 1/\Psi(I)$ and C is the integration constant. Putting (3.9) into (2.4) one gets the equation for the electromagnetic field

$$(e^{-2\gamma}\,\mathcal{A}')' - C^2\, P_I\, e^{-2\gamma}\,\mathcal{A} = 0, \tag{3.10}$$

where the second term could be naturally interpreted as the induced nonlinearity. In view of (3.9) one rewrites the Einstein equation (3.4) and the result of adding the equations (3.3) and (3.4) as follows :

$$\gamma'^2 = -G\,(C^2\, P - \mathcal{A}'^2\, e^{-2\gamma}) + K, \qquad K = k^2\,\mathrm{sign} k, \tag{3.11}$$

$$\gamma'' = G\, e^{-2\gamma}\,(\mathcal{A}'^2 + C^2\,\mathcal{A}^2\, P_I). \tag{3.12}$$

One can easily check that the equation (3.11) is the first integral of the equations (3.10) and (3.12).

Eliminating the term $(P_I\,\mathcal{A})$ between (3.10) and (3.12) one gets the equation:

$$\gamma'' = G\,(\mathcal{A}\,\mathcal{A}'\, e^{-2\gamma})', \tag{3.13}$$

with the evident first integral:

$$\gamma' = G\,\mathcal{A}\,\mathcal{A}'\, e^{-2\gamma} + C_1, \quad C_1 = \text{const}. \tag{3.14}$$

Let us consider the simple case $C_1 = 0$. Then from (3.14) we get

$$e^{2\gamma} = G\,\mathcal{A}^2 + H, \qquad H = \text{const}. \tag{3.15}$$

Substituting γ' and $e^{2\gamma}$ from (3.14) and (3.15) into (3.11), we find for $\mathcal{A}(\xi)$ the differential equation:

$$\mathcal{A}'^2 \,(G\,\mathcal{A}^2 \,+\, H)^{-2} \;=\; (G\,C^2\,P \,-\, K)/\,G\,H, \tag{3.16}$$

which can be solved by quadrature:

$$\int \frac{d\mathcal{A}}{(G\,\mathcal{A}^2 \,+\, H)\,\sqrt{G\,C^2\,P \,-\, K}} \;=\; \pm\,(1/\sqrt{G\,H})\,(\xi \,-\, \xi_0), \quad \xi_0 \;=\; \text{const.} \tag{3.17}$$

It is clear that the configuration obtained has a center if and only if $e^\beta \;=\; 0$ at some $\xi \;=\; \xi_c$. One can show [4] that the conditions for the center $\xi_c \;=\; \infty$ to be regular imply $K \;=\; 0$ and the following behavior of the field quantities in the vicinity of the point $\xi_c \;=\; \infty$:

$$\gamma' \;=\; O\,(\xi^{-2}), \quad \mathcal{A}' \to \mathcal{A}_c \neq \infty, \quad \mathcal{A}' \to 0, \tag{3.18}$$

$$\xi^4\,P(I) \to 0, \quad |\,\xi^4\,I\,P_I\,| < \infty. \tag{3.19}$$

In view of (3.18) we deduce from (3.14) that $C_1 \;=\; 0$ in accordance with the earlier supposition.

Now we can write the boundary conditions on the surface of the critical sphere $\xi \;=\; \xi_0$:

$$T^\nu_\mu \;=\; \mathcal{A} \;=\; \mathcal{A}' \;=\; 0, \quad e^\gamma \;=\; 1, \quad e^\beta \;=\; 1/\xi_0 \;>\; 0. \tag{3.20}$$

Due to (3.20) and (3.15) we infer that $H = 1$.

Let us now choose the function $P(I)$ as follows:

$$P(I) \;=\; J^{(1\,-\,2/\sigma)}\,[(1 \,-\, J)^{1/\sigma} \,-\, J^{1/\sigma}]^2\,(1 \,-\, J), \tag{3.21}$$

where $J \;=\; G\,I$; $\sigma \;=\; 2n \,+\, 1$; $n = 1, 2, 3 \cdots$ Then on account of $K \;=\; 0$ and $H \;=\; 1$ we get from (3.17) the following expression for $\mathcal{A}(x)$:

$$\mathcal{A}(\xi) \;=\; (1/\sqrt{G})\,[1 \,-\, \exp\,(-\frac{2\,C\,\sqrt{G}}{\sigma}\,(\xi \,-\, \xi_0))]^{\sigma/2}. \tag{3.22}$$

As one can see from (3.22), the conditions (3.18) and (3.19) for the center to be regular and the matching conditions (3.20) on the surface of the critical sphere are fulfilled if $\sigma \;>\; 2$.

Recalling that $J \;=\; G\,\mathcal{A}^2/\,(G\,\mathcal{A}^2 \,+\, 1)$, we get from (3.22) that $J(\infty) \;=\; 1/2$ and $J(\xi_0) \;=\; 0$, thus implying:

$$P(I)\,|_{\xi\,=\,\infty} =\; P(I)\,|_{\xi\,=\,\xi_0} =\; 0. \tag{3.23}$$

It means that at $\xi \;=\; \xi_c$ and $\xi \;=\; \xi_0$, the interaction function $\Psi(I) \;=\; 1/P(I)$ is singular. It turns out nevertheless that the energy density T^0_0 is regular at these points due to the fact that it contains $\Psi(I)$ as a multiplier in the form:

$$e^{-2\alpha}\,\varphi'^2\,\Psi \;=\; C^2\,e^{-2\alpha}\,P(I), \tag{3.24}$$

which tends to zero as $\xi \;\to\; \xi_c$ or $\xi \;\to\; \xi_0$.

As follows from (3.22), for the limiting case $\xi_0 = 0$, when the critical sphere goes to the spatial infinity and the solution in question is defined at $0 \le \xi \le \infty$, it appears that at spatial infinity ($\xi = 0$) $\mathcal{A} = 0$ and $P(I) = 0$. In this case we obtain the usual soliton-like configuration not possessing any sharp boundary.

Note that at spatial infinity ($\xi = 0$) one can compare the metric found with the Schwarzschild one and the electrical field with the Coulomb one, thus determining the total mass m and the charge q of the system:

$$G\,m = -\,\gamma'(0), \qquad q = -\,\mathcal{A}'(0).$$

Taking into account that $e^{2\gamma} = G\,\mathcal{A}^2 + 1$, one can find through the use of (3.22) that for $\xi_0 = 0$, $\mathcal{A}'(0) = -q = 0$ and $\gamma'(0) = -G\,m = 0$. Therefore, the total energy of the soliton-like system,defined as the sum of the material fields energy and that of the gravitational field, vanishes. If now one chooses the integration constant $\xi_0 > 0$, then the field configuration with the sharp boundary (droplet) appears. In this case for $\xi \le \xi_0$ one obtains $\mathcal{A}(\xi) = 0$ and $e^{2\gamma} = 1$, i.e. outside of the droplet gravitational and electromagnetic fields disappear, that implies the vanishing of the total mass and the charge of the system. This unusual property makes the droplet-like object poorly visible for the outer observer.

Let us now calculate the matter field energy density:

$$T_0^0 = (C^2/\,8\,\pi)\,e^{-2\alpha}\,[\,P\,(1 + e^{2\gamma}) + 2\,I\,P_I(I)\,]. \tag{3.25}$$

To this end we substitute into (3.25) the expression for $e^{-2\alpha} = \xi^4\,(G\,A^2 + 1)$, $P(I)$ and $\mathcal{A}(\xi)$ using (3.21) and (3.22). It should be emphasized that the field energy is localized in a small region ($\xi_0 \le \xi < \infty$) :

$$T_0^0(\xi)\;|_{\xi\to\infty} \to 0, \qquad T_0^0(\xi)\;|_{\xi\to\xi_0} \to 0, \tag{3.26}$$

namely, inside the critical sphere with the radius

$$R = \int\limits_0^\infty d\xi\, e^{\alpha(\xi)} < \infty.$$

One can readily derive from (3.25) the energy E_f of the matter fields:

$$E_f = \int d^3x\,\sqrt{-^3\,g}\,T_0^0 = (C/2)\int\limits_0^{1/\sqrt{G}} d\mathcal{A}\,e^{-3\gamma}\,[\sqrt{P}(1 + e^{2\gamma}) + 4\,I\,(\sqrt{P})_I\,], \tag{3.27}$$

Using (3.21) and the expressions

$$e^{2\gamma} = 1/(1 - J), \qquad \mathcal{A} = \sqrt{G\,P\,(1 - J)},$$

one finds from (3.27) after integrating by parts,

$$E_f = (C/4\sqrt{G})\int\limits_0^{1/2}[(2 - J)\sqrt{J\,P}/(1 - J) + 2\,P_J\,\sqrt{J^3/P}]\,dJ =$$

$$= (C/4\sqrt{G}) \int\limits_0^{1/2} [(5\,J \,-\, 4)\,\sqrt{J\,P}/(1 \,-\, J)]\,dJ \,<\, 0.$$

Knowing that the total energy of the droplet-like object is zero this inequality implies the positivity of its gravitational energy.

In order to clarify the fact that the role of the gravitational field in forming the droplet-like configuration is not decisive it is worthwhile to compare the solution obtained with that in the flat space-time, described by the interval

$$ds^2 \,=\, dt^2 \,-\, dr^2 \,-\, r^2\,[d\theta^2 \,+\, \sin^2\theta\, d\phi^2].$$

In the latter case the equation (2.3) admits the solution

$$\varphi'(r) \,=\, C\,P(r)/r^2. \tag{3.28}$$

Substituting (3.28) into (2.4), one finds that the equation for the electromagnetic field can be solved by quadrature:

$$\int d\mathcal{A}/\sqrt{P} \,=\, \pm\, C\,(\frac{1}{r} \,-\, \frac{1}{r_0}), \quad r_0 \,=\, \mathrm{const}. \tag{3.29}$$

Note that the droplet-like configuration $\mathcal{A}(r)$ will be similar to (3.22) if one chooses the function $P(I)$ more simple than (3.21):

$$P(I) \,=\, J^{1\,-\,2/\sigma}\,(1 \,-\, J^{1/\sigma})^2, \qquad J \,=\, \lambda\,I, \tag{3.30}$$

where $\lambda \,=\, \mathrm{const};\quad \sigma \,=\, 2\,n \,+\, 1;\quad n \,=\, 1, 2, 3, \cdots$. Then substituting (3.30) into (3.29) one gets the solution

$$\mathcal{A}(r) \,=\, (1/\sqrt{\lambda})\,[1 \,-\, \exp(-\frac{2\,C\,\lambda}{\sigma}\,(\frac{1}{r} \,-\, \frac{1}{r_0}))]^{\sigma/2}. \tag{3.31}$$

One can see from (3.31) that $\mathcal{A}(r) \,=\, 0$ as $r \,\geq\, r_0$, i.e. the charge of the flat space-time droplet configuration also vanishes. For this solution the regularity conditions at the center $r \,=\, 0$ and on the surface of the critical sphere $r \,=\, r_0$ are evidently fulfilled. It similarly appears that for $r \,=\, \infty$ one finds the usual soliton-like structure with field vanishing as $r \,\to\, \infty$. The field energy E_f is defined as follows:

$$E_f \,=\, C \int\limits_{\mathcal{A}(r_0)}^{\mathcal{A}(0)} d\mathcal{A}\,(\sqrt{P} \,+\, I\,P_I/\sqrt{P}) \,=\, C\sqrt{P\,I}\,|_{\mathcal{A}(r_0)}^{\mathcal{A}(0)}\,. \tag{3.32}$$

Inspecting that $P\,I \,=\, 0$ both at $r \,=\, 0$ and $r \,=\, r_0$, we arrive through (3.32) at $E_f \,=\, 0$.

Thus in the flat space-time as well as for the self-gravitating system, the total energy and charge of the droplet-like configuration vanish.

4 Configurations with cylindrical symmetry

Let us now search for static cylindrically-symmetric solutions to the equations (2.2)-(2.4). In this case the metric can be chosen as follows [5]:

$$ds^2 = e^{2\gamma(x)}\, dt^2 - e^{2\alpha(x)}\, dx^2 - e^{2\beta(x)}\, d\phi^2 - e^{2\mu(x)}\, dz^2. \tag{4.1}$$

In (4.1) all the components of the metrical tensor depend on the single spatial coordinate $x \in [x_0, x_a]$, where x_a is the value of x on the axis of symmetry, defined by the condition $\exp[\beta(x_a)] = 0$, and x_0 is the value of x on the surface of the critical cylinder. The coordinates z and ϕ take their standard values: $z \in [-\infty, \infty]$, $\phi \in [0, 2\pi]$. As in the previous case, the electromagnetic field is described by the time component of the 4-potential $\mathcal{A}_0(x) = \mathcal{A}(x)$ and by the component $\mathcal{F}_{1\,0} = d\mathcal{A}/dx = \mathcal{A}'$ of the field strength tensor .

Later on it will be convenient to use the coordinate condition [5]:

$$\alpha = \beta + \mu + \gamma,$$

that permits to present the system of the Einstein equations in the form:

$$\mu'' + \beta'' - V = -\,\kappa\, T^0_0\, e^{2\alpha}, \tag{4.2}$$

$$\mu'\,\beta' + \beta'\,\gamma' + \gamma'\,\mu' = V = -\kappa\, T^1_1\, e^{2\alpha}, \tag{4.3}$$

$$\gamma'' + \beta'' - V = -\,\kappa\, T^2_2\, e^{2\alpha}, \tag{4.4}$$

$$\mu'' + \gamma'' - V = -\,\kappa\, T^3_3\, e^{2\alpha}, \tag{4.5}$$

As in the preceding section, the energy-momentum tensor of interacting fields is also defined by the equations (3.6), (3.7).

Adding together the equations (4.3) and (4.4) and using (3.7), one obtains the simple equation:

$$\gamma'' + \beta'' = 0, \tag{4.6}$$

with the solution

$$\beta(x) + \gamma(x) = C_2\, x, \qquad C_2 = \text{const.} \tag{4.7}$$

Notice that the second integration constant in (4.7) can be taken trivial, as it determines only the choice of scale.

In a similar way the addition of equations (4.3) and (4.5) leads to the equation:

$$\gamma'' + \mu'' = 0, \tag{4.8}$$

with the solution

$$\mu(x) + \gamma(x) = C_3\, x, \qquad C_3 = \text{const.} \tag{4.9}$$

Solving the equation (2.2) in the metric (4.1), one gets the same result as in (3.9), i.e.

$$\varphi'(x) = C\, P(I). \tag{4.10}$$

Substituting (4.10) into (2.4), one finds the equation for the electromagnetic field, coincident with (3.10).

Now as in the previous case, we use the equation (4.3) and sum of equations (4.2) and (4.3) which in view of (4.6) and (4.8), take the form:

$$\gamma'^2 - C_2 C_3 = -G (C^2 P - \mathcal{A}'^2 e^{-2\gamma}), \tag{4.11}$$

$$\gamma'' = G e^{-2\gamma} (\mathcal{A}'^2 + C^2 \mathcal{A}^2 P_I). \tag{4.12}$$

Elimination of $P_I \mathcal{A}$ between the equations (3.10) and (4.12) gives the equation (3.13) with the solution (3.14). Integrating (3.14) under the choice $C_1 = 0$, one obtains the equality (3.15). Finally, substituting γ' from (3.14) and $e^{2\gamma}$ from (3.15) into (4.11), one gets the equation for $\mathcal{A}(x)$:

$$\mathcal{A}'^2 (G \mathcal{A}^2 + H)^{-2} = (G C^2 P - C_2 C_3)/ G H. \tag{4.13}$$

The equation (4.13) can be solved by quadrature:

$$\int \frac{d\mathcal{A}}{(G \mathcal{A}^2 + H) \sqrt{G C^2 P - C_2 C_3}} = \pm (1/\sqrt{G H}) (x - x_0). \tag{4.14}$$

Let us formulate regularity conditions to be satisfied by the solutions to the equations (2.2)-(2.4) on the axis of symmetry defined by the value $x = x_a$, where $\beta(x_a) = -\infty$. Choosing in (4.7) $C_2 < 0$ and taking into account that for the regular solutions $\mid \gamma \mid < \infty$, we get $x_a = \infty$. The regularity conditions are similar to (3.18) and (3.19) for the case of spherical symmetry, implying that the following relations hold as $x \to x_a = \infty$:

$$\gamma' \to 0, \quad \mathcal{A}' \to \mathcal{A}_c \neq \infty, \quad \mathcal{A}' \to 0, \tag{4.15}$$

$$e^{2|C_2| x} P(I) \to 0, \quad e^{2|C_2| x} \mid I P_I \mid < \infty, \quad C_3 \neq 0. \tag{4.16}$$

(53) Boundary conditions on the surface of the critical cylinder $x = x_a$ can be written as follows:

$$T^{\nu}_{\mu} = \mathcal{A} = \mathcal{A}' = 0, \quad e^{\gamma} = 1, \quad e^{\beta} = e^{-|C_2| x} > 0. \tag{4.17}$$

The conditions (4.17) together with the relations $e^{2\gamma} = G \mathcal{A}^2 + H$, imply that $H = 1$.

Choosing $P(I)$ in the form (3.21), one can find the expression for $\mathcal{A}(x)$ which is similar to (3.22):

$$\mathcal{A}(x) = (1/\sqrt{G}) [1 - \exp (-\frac{2 C \sqrt{G}}{\sigma} (x - x_0))]^{\sigma/2}. \tag{4.18}$$

As one can readily see from (4.18), the conditions (4.15), (4.16) and (4.17) are fulfilled if $\mid C_2 \mid \leq C \sqrt{G}/\sigma$. It is noteworthy that at $x \leq x_0$, $\mathcal{A}(x) \equiv 0$ and the space-time is flat, the gravitational field being absent .

There is a principal difference between solutions (3.22) and (4.18). For the case of spherical symmetry the droplet-like solution can be transformed to the soliton-like one if the boundary ξ_0 is removed by putting $\xi_0 = 0$ (as in this case $\exp[\beta(\xi_0)] = 1/\xi_0 = \infty$). On the contrary, for the case of cylindrical symmetry the removal of the boundary is equivalent to putting $x_0 = -\infty$, as in this case $\exp[\beta(x_0)] = \exp(- \mid C_2 \mid x_0) = \infty$. Under this last choice the solution (4.18) takes constant value $\mathcal{A}(x) = 1/\sqrt{G}$ and the soliton structure disappears. For the considered case, as well as for that of spherical symmetry, the density of the field energy is given by equation (3.25) and the linear density of energy is similar to (3.27):

$$E_f = (C/4) \int\limits_0^{1/\sqrt{G}} d\mathcal{A}\, e^{-3\gamma} [\sqrt{P}(1 + e^{2\gamma}) + 4\, I\, (\sqrt{P})_I], \qquad (4.19)$$

Substituting $P(I)$ from (3.21) into (4.19), one can find that E_f is finite and the total energy $E_f + E_g$ turns out to be zero.

Now it is worthwhile to make again the comparison with the flat-space solutions of the equations (2.3) and (2.4), using the interval:

$$ds^2 = dt^2 - d\rho^2 - \rho^2\, d\phi^2 - dz^2.$$

In this case the scalar field equation (2.3) admits the solution:

$$\varphi'(\rho) = C\, P(I)/\rho, \quad P(I) = 1/\Psi(I), \quad C = \text{const.} \qquad (4.20)$$

Inserting (4.20) into (2.4), one can find the electromagnetic field equation which admits the solution in quadratures:

$$\int \frac{d\mathcal{A}}{\sqrt{P(I)}} = \pm C \ln \frac{\rho}{\rho_0}, \quad \rho_0 = \text{const.} \qquad (4.21)$$

Substituting $P(I)$ from (3.30) in (4.21), one gets the solution of the droplet-like form:

$$\mathcal{A}(\rho) = (1/\sqrt{\lambda}\, [1 - (\frac{\rho}{\rho_0})^{2\, C\, \sqrt{\lambda}/\sigma}]^{\sigma/2}. \qquad (4.22)$$

One concludes from (4.22) that $\mathcal{A}(\rho \ge \rho_0) \equiv 0$. It means that the electric charge of the system is zero . For the solution (4.22) the regularity conditions both on the axis $\rho = 0$ and on the surface of the critical cylinder $\rho = \rho_0$ are fulfilled if $C\sqrt{\lambda} \ge \sigma$. It is noteworthy that in the case of cylindrical symmetry, both in the flat space-time and with account of the proper gravitational field, there do not exist any soliton-like solutions, as for the choice $\rho_0 = \infty$ the solution (4.22) degenerates into the constant: $\mathcal{A}(\rho) = 1/\sqrt{\lambda}$.

The linear density of the field energy in flat space-time can be found from the expression similar to (3.32), and as well as in the case of spherical symmetry, it is equal to zero:

$$E_f = \frac{C}{2} \sqrt{P\, I}\, |_{\mathcal{A}(\rho_0)}^{\mathcal{A}(0)} = 0.$$

5 Conclusion

In conclusion, we underline once more the principal difference between the droplet-like solutions with spherical symmetry and those with cylindrical one. In the first case there exists a possibility of continuous transformation of the droplet-like configuration into the solitonian one by transporting the sharp boundary to the infinity. As for the second case, there is no such a possibility, and the soliton-like configuration disappears when the boundary is smoothed tending to the infinity.

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