PACS 04.20 Jb

Exact Self-Consistent Solutions to Nonlinear Spinor Field Equations in Bianchi Type-I Space-Time[1]

Yu.P.Rybakov⋆, B.Saha†, G.N.Shikin⋆.
⋆ Russian Peoples' Friendship University, Moscow,
† Joint Institute for Nuclear Research, Dubna, Moscow region.

Summary

Self-consistent solutions to nonlinear spinor field equations in General Relativity are studied for the case of Bianchi type-I space-time. It should be emphasized the absence of initial singularity for some types of solutions and also the isotropic mode of space-time expansion in some special cases.

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The aim of the paper is to find some exact self-consistent solutions to the spinor field equations with nonlinear terms being arbitrary functions of the invariant $S = \bar\psi\psi$, for Bianchi type-I space-time. Equations with power nonlinearity in spinor field Lagrangian $L_N = \lambda S^n$, where λ is the coupling constant , have been thoroughly studied. In this case it is shown that indicated equations for $n > 2$ possess solutions both regular and singular at the initial moment of time. Singularity remains absent for the case of field system with broken dominant energy condition. It is also shown that if in the spinor field equation the massive parameter $m \neq 0$ and $n \geq 2$ then at $t \to \infty$ isotropization of Bianchi type-I space-time expansion takes place, while for $m = 0$ the expansion is anisotropic. Properties of solutions to the spinor field equation for $1 < n < 2$ and $0 < n < 1$ we also studied. It was found that in these cases there does not exist solution, which is regular at initial moment of time. At $t \to \infty$ the isotropization process of Bianchi type-I space-time expansion takes place both for $m \neq 0$ and for $m = 0$.

The Lagrangian for the self-consistent system of spinor and gravitation fields can be written as

$$L = \frac{R}{2\kappa} + \frac{i}{2}\Big[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi\Big] - m\bar\psi\psi + L_N \tag{1}$$

with R- being the scalar curvature, κ- being the Einstein's gravitational constant, $L_N = F(S)$, being an arbitrary function of $S = \bar\psi\psi$,. Bianchi type-I space-time metric can be chosen in the form [1]

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2, \tag{2}$$

From Lagrangian (1) we will get Einstein equations, spinor field equations and components of energy-momentum tensor for the spinor field.

We will use Einstein equations for $a(t), b(t)$ and $c(t)$ in the form [1]:

$$\frac{\ddot a}{a} + \frac{\dot a}{a}\Big(\frac{\dot b}{b} + \frac{\dot c}{c}\Big) = -\kappa\Big(T_1^1 - \frac{1}{2}T\Big), \tag{3}$$

$$\frac{\ddot b}{b} + \frac{\dot b}{b}\Big(\frac{\dot a}{a} + \frac{\dot c}{c}\Big) = -\kappa\Big(T_2^2 - \frac{1}{2}T\Big), \tag{4}$$

$$\frac{\ddot c}{c} + \frac{\dot c}{c}\Big(\frac{\dot a}{a} + \frac{\dot b}{b}\Big) = -\kappa\Big(T_3^3 - \frac{1}{2}T\Big), \tag{5}$$

[1] "Communications in Theoretical Physics", **3** (2), pp. 199-210 (1994).

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}=-\kappa\Big(T_0^0-\frac{1}{2}T\Big), \tag{6}$$

where point means differentiation with respect to t, and $T=T_\mu^\mu$. Nonlinear spinor field equations and components of its energy-momentum tensor can be written as follows:

$$i\gamma^\mu\nabla_\mu\psi-m\psi+L'_N\psi=0, \qquad L'_N\psi:=\frac{dL_N}{d\bar{\psi}} \tag{7}$$

$$T_\mu^\rho=\frac{i}{4}g^{\rho\nu}\Big(\bar{\psi}\gamma_\mu\nabla_\nu\psi+\bar{\psi}\gamma_\nu\nabla_\mu\psi-\nabla_\mu\bar{\psi}\gamma_\nu\psi-\nabla_\nu\bar{\psi}\gamma_\mu\psi\Big)-\delta_\mu^\rho L \tag{8}$$

while L on account of spinor field equations takes the form:

$$\Big[\frac{1}{2}\Big(\bar{\psi}\frac{\partial L_N}{\partial\bar{\psi}}+\frac{\partial L_N}{\partial\psi}\psi\Big)-L_N\Big].$$

In (7) and (9) ∇_μ denotes the covariant derivative of spinor, having the form [2]:

$$\nabla_\mu\psi=\frac{\partial\psi}{\partial x^\mu}-\Gamma_\mu\psi, \tag{9}$$

where $\Gamma_\mu(x)$ are spinor affine connection matrices. $\gamma^\mu(x)$ matrices are defined for the metric (2) as follows. Using the equality

$$g_{\mu\nu}(x)=e_\mu^a(x)e_\nu^b(x)\eta_{ab}, \quad \gamma_\mu(x)=e_\mu^a(x)\bar{\gamma}_a,$$

where $\eta_{ab}=\mathrm{diag}(1,-1,-1,-1)$, $\bar{\gamma}_a$ being flat space-time Dirac matrices, e_μ^a denoting a set of tetrad 4-vectors, we will get

$$\gamma^0=\bar{\gamma}^0, \quad \gamma^1=\bar{\gamma}^1/a(t), \quad \gamma^2=\bar{\gamma}^2/b(t), \quad \gamma^3=\bar{\gamma}^3/c(t).$$

$\Gamma_\mu(x)$ matrices are defined by the equality

$$\Gamma_\mu(x)=\frac{1}{4}g_{\rho\sigma}(x)\Big(\partial_\mu e_\delta^b e_b^\rho-\Gamma_{\mu\delta}^\rho\Big)\gamma^\sigma\gamma^\delta,$$

which gives

$$\Gamma_0=0, \quad \Gamma_1=\frac{1}{2}\dot{a}(t)\bar{\gamma}^1\bar{\gamma}^0, \quad \Gamma_2=\frac{1}{2}\dot{b}(t)\bar{\gamma}^2\bar{\gamma}^0, \quad \Gamma_3=\frac{1}{2}\dot{c}(t)\bar{\gamma}^3\bar{\gamma}^0, \tag{10}$$

Flat space-time matrices we will choose in the form, given in [3].

We will study the space-independent solutions to spinor field equation (7) so that $\psi=V(t)$. In this case equation (7) together with (9) and (10) can be written as:

$$i\bar{\gamma}^0\Big(\frac{\partial}{\partial t}+\frac{\dot{\tau}}{2\tau}\Big)V-(m-L'_N)V=0, \quad \tau(t)=a(t)b(t)c(t). \tag{11}$$

For the components $\psi_\rho=V_\rho(t)$, $\rho=1,2,3,4$, from (11) one deduces the following system of equations:

$$\begin{aligned}\dot{V}_r+\frac{\dot{\tau}}{2\tau}V_r+i(m-F_1)V_r &= 0, \quad r=1,2;\\ \dot{V}_l+\frac{\dot{\tau}}{2\tau}V_l-i(m-F_1)V_l &= 0, \quad l=3,4.\end{aligned} \tag{12}$$

where $F_1 := dF/dS$. From (12) we will find the equation for invariant function

$$S = \bar{\psi}\psi = V_1^* V_1 + V_2^* V_2 - V_3^* V_3 - V_4^* V_4 :$$

$$\dot{S} + \frac{\dot{\tau}}{\tau} S = 0, \tag{13}$$

which leads to

$$S = \frac{C_0}{\tau}, \quad C_0 = \text{const.} \tag{14}$$

As in the considered case F depends only on S, from (14) it follows that $F(S)$ and $F_1(S)$ are functions of $\tau = abc$. Taking this fact into account, integration of the system of equations (12) leads to the expressions

$$\begin{aligned} V_r(t) &= \frac{C_r}{\sqrt{\tau}} \exp\Big[-i(mt - \int F_1 dt)\Big], \qquad r = 1,2; \\ V_l(t) &= \frac{C_l}{\sqrt{\tau}} \exp\Big[i(mt - \int F_1 dt)\Big], \qquad l = 3,4. \end{aligned} \tag{15}$$

where C_r and C_l are integration constants. Putting (15) into (8), we will get the following expressions for the components of the energy-momentum tensor for the spinor field:

$$T_0^0 = \frac{i}{2} N + R, \quad T_1^1 = T_2^2 = T_3^3 = R, \quad T = T_\alpha^\alpha = \frac{i}{2} N + 4R, \tag{16}$$

where

$$N = -\frac{2i}{\tau}(C_1^2 + C_2^2 - C_3^2 - C_4^2)(m - F_1), \quad R = F_1(S) S - F(S),$$

$$C_1^2 + C_2^2 - C_3^2 - C_4^2 C = C_0.$$

Summation of Einstein equations (3),(4) and (5) leads to the equation

$$\frac{\ddot{\tau}}{\tau} = -\kappa(T_1^1 + T_2^2 + T_3^3 - \frac{3}{2} T) = 3\kappa(\frac{i}{4} N + R) \tag{17}$$

As the right hand side of the equation (17) is the function of S or $\tau(t) = abc$, this equation takes the form

$$\ddot{\tau} + \Phi(\tau) = 0 \tag{18}$$

Equation (18) possesses exact solutions for arbitrary function $\Phi(\tau)$ [4]. Giving the explicit form of $L_N = F(S)$, from (18) one can find concrete function $\tau(t) = abc$. Putting the obtained function in (15), one can get expressions for components $V_\alpha(t), \quad \alpha = 1,2,3,4$.

Let us express a, b, c through τ. For this we notice that subtraction of Einstein equations (3)-(4) leads to the equation

$$\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{c}}{ac} - \frac{\dot{b}\dot{c}}{bc} = \frac{d}{dt}\Big(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\Big) + \Big(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\Big)\Big(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\Big) = 0. \tag{19}$$

Equation (19) possesses the solution

$$\frac{a}{b} = D_1 \exp\Big(X_1 \int \frac{dt}{\tau}\Big), \quad D_1 = \text{const.}, \quad X_1 = \text{const.} \tag{20}$$

Subtracting equations (3)-(5) and (4)-(5) one finds the equations similar to (19), having solutions

$$\frac{a}{c} = D_2 \exp\Big(X_2 \int \frac{dt}{\tau}\Big), \quad \frac{b}{c} = D_3 \exp\Big(X_3 \int \frac{dt}{\tau}\Big), \tag{21}$$

where D_2, D_3, X_2, X_3 are integration constants. There is a functional dependence between the constants D_1, D_2, D_3, X_1, X_2, X_3:

$$D_2 = D_1\, D_3, \qquad X_2 = X_1\, +\, X_3.$$

Using the equations (19),(20) and (21), we rewrite $a(t), b(t), c(t)$ in the explicit form:

$$\begin{aligned}
a(t) &= (D_1^2 D_3)^{\frac{1}{3}}\tau^{\frac{1}{3}}\exp\Bigl[\frac{2X_1+X_3}{3}\int\limits_{t_0}^{t}\tau^{-1}dt'\Bigr]\\
b(t) &= (D_1^{-1} D_3)^{\frac{1}{3}}\tau^{\frac{1}{3}}\exp\Bigl[-\frac{X_1-X_3}{3}\int\limits_{t_0}^{t}\tau^{-1}dt'\Bigr]\\
c(t) &= (D_1 D_3^2)^{-\frac{1}{3}}\tau^{\frac{1}{3}}\exp\Bigl[-\frac{X_1+2X_3}{3}\int\limits_{t_0}^{t}\tau^{-1}dt'\Bigr] \qquad (22)
\end{aligned}$$

where t_0 is the initial time.

Thus the previous system of Einstein equations and nonlinear spinor field ones is completely integrated. In this process of integration only first three of the complete system of Einstein equations have been used. General solutions to these three second order equations have been obtained. The solutions contain six arbitrary constants: D_1, D_3, X_1, X_3 and two others, that were obtained while solving equation (18). Equation (6) is the consequence of first three of Einstein equations. To verify the correctness of obtained solutions, it is necessary to put a, b, c in (6). It should lead either to identity or to some additional constraint between the constants. Putting a, b, c from (22) in (6) one can get the following equality:

$$\frac{1}{3\tau}\Bigl[3\ddot\tau - 2\frac{\dot\tau^2}{\tau} + \frac{2}{3\tau}\Bigl(X_1^2 + X_1X_3 + X_3^2\Bigr)\Bigr] = -\kappa\Bigl(T_0^0 - \frac{1}{2}T\Bigr), \qquad (23)$$

that guaranties the correctness of obtained solutions.

Let us consider the concrete type of nonlinear spinor field equation, when $L_N = F(S) = \lambda S^n, \lambda$ - being the coupling constant, $n > 1$. In this case we will get the following expressions for the energy-momentum tensor components of the spinor field:

$$N = -2iS\Bigl[m - \lambda n S^{n-1}\Bigr], \qquad R = \lambda(n-1)S^n,$$

$$T_0^0 = \frac{i}{2}N + R = mS - \lambda S^n, \qquad T_1^1 = T_2^2 = T_3^3 = R = \lambda(n-1)S^n,$$

$$T = T_\alpha^\alpha = mS + (3n-4)\lambda S^n, \quad T_0^0 - \frac{1}{2}T = \frac{1}{2}mS + \frac{2-3n}{2}\lambda S^n, \qquad (24)$$

$$T_1^1 - \frac{1}{2}T = T_2^2 - \frac{1}{2}T = T_3^3 - \frac{1}{2}T = -\frac{1}{2}mS - \frac{n-2}{2}\lambda S^n. \qquad (25)$$

Putting (25) in (17) one can get the equation for $\tau = abc = \frac{C_0}{S}$:

$$\ddot\tau = \frac{3}{2}\kappa C_0\Bigl[m + \lambda(n-2)\frac{C_0^{n-1}}{\tau^{n-1}}\Bigr]. \qquad (26)$$

We will first study the solution to the equation (26) when $\lambda = 0$, i.e. when the spinor field nonlinearity remains absent: $L_N = 0$. The reason to get the solution to the self-consistent system of equations for the linear spinor and gravitational fields is the necessity of comparing this solution with that for the system of equations for the nonlinear spinor and gravitational fields that permits to clarify the role of nonlinear spinor terms in the evolution of the cosmological model in question. In this case the solution to the equation (26) takes form:

$$\tau(t) = \frac{1}{2}Mt^2 + y_1 t + y_0 \tag{27}$$

where $M = \frac{3}{2}\kappa m C_0, \quad C_0 > 0, \quad y_1, y_0 = \text{const}$. Putting $\tau(t)$ from (27) into (15) and (22), we can get explicit expressions for the components of spinor field functions and metric functions a, b, c:

$$V_r(t) = \frac{C_r e^{-imt}}{\frac{1}{2}Mt^2 + y_1 t + y_0}, \qquad V_l(t) = \frac{C_l e^{imt}}{\frac{1}{2}Mt^2 + y_1 t + y_0} \tag{28}$$

$$\begin{aligned}
a(t) &= (D_1^2 D_3)^{\frac{1}{3}}(\frac{1}{2}Mt^2 + y_1 t + y_0)^{\frac{1}{3}} Z^{2(2X_1+X_3)/3B} \\
b(t) &= (D_1^{-1} D_3)^{\frac{1}{3}}(\frac{1}{2}Mt^2 + y_1 t + y_0)^{\frac{1}{3}} Z^{-2(X_1-X_3)/3B} \\
c(t) &= (D_1 D_3^2)^{-\frac{1}{3}}(\frac{1}{2}Mt^2 + y_1 t + y_0)^{\frac{1}{3}} Z^{-2(X_1+2X_3)/3B}
\end{aligned} \tag{29}$$

where

$$Z = \frac{(t-t_1)}{(t-t_2)}, \quad B = M(t_1 - t_2),$$

and $t_{1,2} = -y_1/M \pm \sqrt{(y_1/M)^2 - 2y_0/M}$ are the roots of the quadratic equation $Mt^2 + 2y_1 t + 2y_0 = 0$. Substituting $\tau(t)$ from (27) into (23), one can get the following equality:

$$y_1^2 - 2My_0 = \frac{X_1^2 + X_1 X_3 + X_3^2}{3} \tag{30}$$

which leads to $y_1^2 - 2My_0 > 0$. This means that the quadratic trinomial in (27) possesses real roots, i.e. $\tau(t)$ in (27) turns into zero at $t = t_{1,2}$ and the solution obtained is the singular one.

Let us now study the solutions (27)-(29) at $t \to \infty$. In this case we shall have

$$\tau(t) \approx \frac{3}{4}\kappa m C_0 t^2, \qquad a(t) \approx b(t) \approx c(t) \approx t^{2/3},$$

that leads to the conclusion about the asymptotical isotropization of the expansion process for the initially anisotropic Bianchi type-I space. Thus the solution to the self-consistent system of equations for the linear spinor and gravitational fields is the singular one at the initial time. In the initial state of evolution of the field system the expansion process of space is anisotropic, but at $t \to \infty$ there happens isotropization of the expansion process.

The first integral of equation (26) takes form:

$$\dot{\tau}^2 = 3\kappa C_0 \Big[m\tau - \lambda \frac{C_0^{n-1}}{\tau^{n-2}} + y_2\Big], \quad y_2 = \text{const}. \tag{31}$$

Substituting (26) and (31) into (23) one can get the following relation between integrating constants:

$$y_2 = \frac{X_1^2 + X_1 X_3 + X_3^2}{3} \tag{32}$$

which leads to $y_2 > 0$. Then (31) can be rewritten as:

$$\dot{\tau}^2 = 3\kappa C_0 \Big[m\tau - \lambda \frac{C_0^{n-1}}{\tau^{n-2}} + g^2\Big], \tag{33}$$

where $g^2 = y_2$. The sign C_0 is determined by the positivity of the energy-density T_0^0 of linear spinor field:

$$T_0^0 = \frac{mC_0}{abc} > 0. \tag{34}$$

It is obvious from (34) that $C_0 > 0$. From (33) one can get the solution to the equation (26) in quadratures:

$$\int \frac{\tau^{(n-2)/2} d\tau}{\sqrt{m\tau^{n-1} + g^2\tau^{n-2} - \lambda C_0^{n-1}}} = \sqrt{3\kappa C_0} t \tag{35}$$

The constant of integration in (35) has been taken zero, as it only gives the shift of the initial time. Let us study the properties of solution to equation (26) for $n > 2$, on the base of (35). As $t \to \infty$, from (33) one can get

$$\tau(t) \approx \frac{3}{4}\kappa m C_0 t^2 \tag{36}$$

which coincides with the solution to the linear spinor field equation (27) at $t \to \infty$. It leads to the conclusion about isotropization of the expansion process of the Bianchi type-I space. It should be remarked that the isotropization takes place only if the spinor field equation contains the massive term [cf. the parameter m in (30)]. If m=0 the isotropization does not take place. In this case at $t \to \infty$ from (35) we get

$$\tau(t) \approx \sqrt{3\kappa C_0 g^2} t. \tag{37}$$

Substituting (37) into (22) one comes to the conclusion that the functions $a(t), b(t)$ and $c(t)$ are different. Let us consider the properties of solutions to equation (26) when $t \to 0$. For $\lambda < 0$ from (35) we get

$$\tau(t) = \left[\frac{3}{4} n^2 \kappa \lambda C_0^n\right]^{1/n} t^{2/n} \to 0, \tag{38}$$

i.e. solutions are singular. For $\lambda > 0$, from (35) it follows that $\tau = 0$ can be reached for no value of t as in this case the denominator of the integrand in (35) becomes imaginary. It means that for $\lambda > 0$ there exist regular solutions to the previous system of equations. The absence of the initial singularity in the considered cosmological solution appears to be consistent with the violation for $\lambda > 0$, of the dominant energy condition in the Hawking-Penrose theorem [1].

Let us consider the Heisenberg-Ivanenko equation when in (26) n=2 [5]. In this case the equation for $\tau(t)$ does not contain the nonlinear term and its solution coincides with that of the linear equation, having the form (27). With such n chosen the metric functions a, b, c are given by the equality (29), and the spinor field functions are written as follows:

$$V_r = \frac{C_r}{\sqrt{\tau}} e^{-imt} Z^{4i\lambda C_0/B}, \quad V_l = \frac{C_l}{\sqrt{\tau}} e^{imt} Z^{-4i\lambda C_0/B}$$

where

$$\tau(t) = (1/2)Mt^2 + y_1 t + y_0, \quad Z = \frac{t - t_1}{t - t_2}, \quad B = M(t_1 - t_2),$$

and C_r, C_l are constants of integration. As in the linear case, the obtained solution is singular at initial time and asymptotically isotropic as $t \to \infty$.

We will now study the properties of solutions to equation (26) for $1 < n < 2$. In this case it is convenient to present the solution (35) in the form:

$$\int \frac{d\tau}{\sqrt{m\tau - \lambda\tau^{2-n} C_0^{n-1} + g^2}} = \sqrt{3\kappa C_0} t \tag{39}$$

As $t \to \infty$, from (39) we will get the equality (36), leading to the isotropization of the expansion process. If $m = 0$ and $\lambda > 0$, $\quad \tau(t)$ lives on the interval $0 \leq \tau(t) \leq \left(g^2/\lambda C_0^{n-1}\right)^{1/(2-n)}$. If m=0 and $\lambda < 0$, the relation (39) at $t \to \infty$ leads to the equality:

$$\tau(t) \approx \left[\frac{3}{4} n^2 \kappa \lambda C_0^n\right]^{1/n} t^{2/n}. \tag{40}$$

Substituting (40) into (22) and taking into account that at $t \to \infty$

$$\int \frac{dt}{\tau} \approx \frac{n(3\kappa\lambda n^2 C_0^n)^{1/n}}{(n-2)2^{2/n}} t^{-2/n+1} \to 0$$

due to $-2/n + 1 < 0$, we shall obtain

$$a(t) \sim b(t) \sim c(t) \sim [\tau(t)]^{1/3} \sim t^{2/3n} \to \infty. \tag{41}$$

It means that the obtained solution at $t \to \infty$ tends to the isotropic one. In this case the isotropization is provided not by the massive parameter, but by the degree n in the term $L_N = \lambda S^n$. As $t \to 0$, (39) implies

$$\tau(t) \approx \sqrt{3\kappa C_0 g^2} t, \tag{42}$$

whence $\tau(0) = 0$, that is the obtained solution is singular at the initial time. Thus, for $1 < n < 2$ there exist only singular solutions at initial time. At $t \to \infty$ the isotropization of the expansion process of Bianchi type-I space takes place both for $m \neq 0$ and for $m = 0$.

Let us finally study the properties of the solution to the equation (26) for $0 < n < 1$. In this case we will use the solution in the form (39). As now $2 - n > 1$, then with the increasing of $\tau(t)$ in the denominator of the integrand in (39) the second term $\lambda \tau^{2-n} C_0^{n-1}$ increases faster than the first one. Therefore the solution describing the space expansion can be possible only for $\lambda < 0$. In this case at $t \to \infty$, for $m = 0$ as well as for $m \neq 0$, one can get the asymptotic representation (40) of the solution. This solution, as for the choice $1 < n < 2$, provides asymptotically isotropic expansion of the Bianchi type-I space. For $t \to 0$ in this case we shall get only singular solution of the form (42).

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