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Exact Self-Consistent Solutions to the Interacting Spinor and Scalar Field Equations in Bianchi Type-I Space-Time[1]

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Summary

Self-consistent solutions to the system of spinor and scalar field equations in General Relativity are studied for the case of Bianchi type-I space-time. It should be emphasized the absence of initial singularity for some types of solutions and also the isotropic mode of space-time expansion in some special cases.

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The aim of the paper is to find some exact self-consistent solutions to the equations, describing spinor and scalar field system with the interaction Lagrangian $L_{int} = \varphi_{,\alpha}\varphi^{,\alpha}\Phi(S)$, $\Phi(S)$ being arbitrary function of the invariant $S = \bar\psi\psi$ for Bianchi type-I space-time. Equations for partial choice of $\Phi(S)$, while $[\Phi(S)]^{-1} = 1 + \lambda S^n$, λ being the coupling constant, n being some constant, have been thoroughly studied. It is shown that the equations, mentioned, can possess initially regular, as well as singular solutions, depending on the sign of λ, nevertheless singularity remains absent for solutions describing the field system with broken dominant energy condition.

The Lagrangian for the interacting system of spinor, scalar and gravitation fields can be written as:

$$L = \frac{R}{2\kappa} + \frac{i}{2}\Big[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi\Big] - m\bar\psi\psi + \frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha}\Phi(S), \tag{1}$$

with R being the scalar curvature, κ being the Einstein's gravitational constant. Function $\Phi(S) = 1 + \lambda F(S), S = \bar\psi\psi$, describes the interaction between spinor and scalar fields, λ being the interaction parameter. For $\lambda = 0$ the interaction vanishes and $\Phi(S) = 1$. In this case we have the system of fields with minimal coupling.

Bianchi type-I space-time metric can be chosen in the form [1]

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \tag{2}$$

From Lagrangian (1) we will get Einstein equations, spinor and scalar field equations and components of their energy-momentum tensor. We will use Einstein equations for $a(t), b(t)$ and $c(t)$ in the form [1]:

$$\frac{\ddot a}{a} + \frac{\dot a}{a}\Big(\frac{\dot b}{b} + \frac{\dot c}{c}\Big) = -\kappa\Big(T_1^1 - \frac{1}{2}T\Big), \tag{3}$$

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$$\frac{\ddot{b}}{b}+\frac{\dot{b}}{b}\Big(\frac{\dot{a}}{a}+\frac{\dot{c}}{c}\Big)=-\kappa\Big(T_2^2-\frac{1}{2}T\Big), \tag{4}$$

$$\frac{\ddot{c}}{c}+\frac{\dot{c}}{c}\Big(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\Big)=-\kappa\Big(T_3^3-\frac{1}{2}T\Big), \tag{5}$$

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}=-\kappa\Big(T_0^0-\frac{1}{2}T\Big), \tag{6}$$

where point means differentiation with respect to t, and $T=T_\mu^\mu$.

Spinor and scalar field equations and components of its energy-momentum tensor can be written as follows:

$$i\gamma^\mu\nabla_\mu\psi-m\psi+\frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha}\Phi'(S)\psi=0, \quad \Phi'(S)=\frac{d\Phi}{dS}, \tag{7}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\Big(\sqrt{-g}g^{\nu\mu}\varphi_{,\mu}\Phi(S)\Big)=0, \tag{8}$$

$$T_\mu^\rho=\frac{i}{4}g^{\rho\nu}\Big(\bar{\psi}\gamma_\mu\nabla_\nu\psi+\bar{\psi}\gamma_\nu\nabla_\mu\psi-\nabla_\mu\bar{\psi}\gamma_\nu\psi-\nabla_\nu\bar{\psi}\gamma_\mu\psi\Big)+\varphi_{,\mu}\varphi^{,\rho}\Phi(S)-\delta_\mu^\rho L. \tag{9}$$

In (7) and (9) ∇_μ denotes covariant derivative of spinor, having the form [2]:

$$\nabla_\mu\psi=\frac{\partial\psi}{\partial x^\mu}-\Gamma_\mu\psi, \tag{10}$$

where $\Gamma_\mu(x)$ are spinor affine connection matrices. $\gamma^\mu(x)$ matrices are defined for the metric (2) as follows. Using the equality

$$g_{\mu\nu}(x)=e_\mu^a(x)e_\nu^b(x)\eta_{ab}, \quad \gamma_\mu(x)=e_\mu^a(x)\bar{\gamma}_a,$$

where $\eta_{ab}=diag(1,-1,-1,-1)$, $\bar{\gamma}_a$ being flat space-time Dirac matrices, e_μ^a denoting a set of tetrad 4-vectors, we will get

$$\gamma^0=\bar{\gamma}^0, \quad \gamma^1=\bar{\gamma}^1/a(t), \quad \gamma^2=\bar{\gamma}^2/b(t), \quad \gamma^3=\bar{\gamma}^3/c(t).$$

$\Gamma_\mu(x)$ matrices are defined by the equality

$$\Gamma_\mu(x)=\frac{1}{4}g_{\rho\sigma}(x)\Big(\partial_\mu e_\delta^b e_b^\rho-\Gamma_{\mu\delta}^\rho\Big)\gamma^\sigma\gamma^\delta,$$

which gives

$$\Gamma_0=0, \quad \Gamma_1=\frac{1}{2}\dot{a}(t)\bar{\gamma}^1\bar{\gamma}^0, \quad \Gamma_2=\frac{1}{2}\dot{b}(t)\bar{\gamma}^2\bar{\gamma}^0, \quad \Gamma_3=\frac{1}{2}\dot{c}(t)\bar{\gamma}^3\bar{\gamma}^0, \tag{11}$$

Flat space-time matrices we will choose in the form, given in [3].

We will study the space-independent solutions to spinor and scalar field equations (7), (8) so that

$$\psi=V(t), \quad \varphi=\varphi(t).$$

In this case solution to the equation (8) is:

$$\dot{\varphi}(t)=\frac{C}{\tau\Phi(S)}, \quad C=const, \quad \tau(t)=a(t)b(t)c(t). \tag{12}$$

In accordance with (12) the spinor field equation (7) can be written as:

$$i\bar{\gamma}^0\Big(\frac{\partial}{\partial t}+\frac{\dot{\tau}}{2\tau}\Big)V - mV - \frac{C^2}{2\tau^2}P'(S)V = 0, \qquad (13)$$

where $P(S) = 1/\Phi(S)$; $P'(S) = \frac{dP}{dS} = -\frac{\Phi'}{\Phi^2}$. For the components $\psi_\rho = V_\rho(t)$, $\rho = 1,2,3,4$, from (13) one deduces the following system of equations:

$$\dot{V}_r + \frac{\dot{\tau}}{2\tau}V_r + i\Big(m + \frac{C^2P'}{2\tau^2}\Big)V_r = 0, \quad r = 1,2; \qquad (14)$$

$$\dot{V}_l + \frac{\dot{\tau}}{2\tau}V_l - i\Big(m + \frac{C^2P'}{2\tau^2}\Big)V_l = 0, \quad l = 3,4. \qquad (15)$$

From (14) and (15) we will find the equation for invariant function

$$S = \bar{\psi}\psi = V_1^*V_1 + V_2^*V_2 - V_3^*V_3 - V_4^*V_4 :$$

$$\dot{S} + \frac{\dot{\tau}}{\tau}S = 0, \qquad (16)$$

which leads to

$$S = \frac{C_0}{\tau}, \quad C_0 = const. \qquad (17)$$

As in the considered case P depends only on S, from (17) it follows that $P(S)$ and $P'(S)$ are functions of $\tau = abc$. Taking this fact into account, integration of the system of equations (14) and (15) leads to the expressions

$$V_r(t) = \frac{C_r}{\sqrt{\tau}}exp\Big[-i(mt + \int Qdt)\Big], \qquad r = 1,2;$$

$$V_l(t) = \frac{C_l}{\sqrt{\tau}}exp\Big[i(mt + \int Qdt)\Big], \qquad l = 3,4. \qquad (18)$$

where $Q(t) = \frac{C^2P'}{2\tau^2}$, C_r C_l - integration constants.

Putting (18) into (9), we will get the following expressions for the components of the energy-momentum tensor for the interacting spinor and scalar fields

$$T_0^0 = \frac{i}{2}N + \frac{C^2}{\tau^2}P - R, \quad T_1^1 = T_2^2 = T_3^3 = -R, \qquad (19)$$

where

$$N = -\frac{2i}{\tau}(C_1^2 + C_2^2 - C_3^2 - C_4^2)(m+Q) = -\frac{2iC_0}{\tau}\Big(m + \frac{C^2P'}{2\tau^2}\Big); \qquad (20)$$

$$R = \frac{C^2}{2\tau^2}\Big(P + \frac{C_0P'}{\tau}\Big), \quad T = T_\alpha^\alpha = \frac{i}{2}N + \frac{C^2}{\tau^2}P - 4R. \qquad (21)$$

Summation of Einstein equations (3),(4) and (5) leads to the equation

$$\frac{\ddot{\tau}}{\tau} = -\kappa\Big(T_1^1 + T_2^2 + T_3^3 - \frac{3}{2}T\Big) = 3\kappa\Big(\frac{mC_0}{2\tau} - \frac{C_0C^2P'}{4\tau^3}\Big). \qquad (22)$$

The first integral of the equation (22) takes the form:

$$\dot{\tau}^2 = 3\kappa(mC_0\tau + \frac{1}{2}C^2P + C_1), \quad C_1 = const. \qquad (23)$$

Final solution of the equation (22) reads

$$\int \frac{d\tau}{\sqrt{(mC_0\tau + \frac{1}{2}C^2P + C_1)}} = \pm\sqrt{3\kappa}(t+t_0), \quad t_0 = const. \tag{24}$$

Giving the explicit form of $\Phi(S)$, i.e. $P = 1/\Phi$, from (24) one can find concrete function $\tau(t) = a(t)b(t)c(t)$. Putting the obtained function in (18), one can get expressions for components of spinor function $V_\rho(t)$, where $\rho = 1, 2, 3, 4$.

Let us express a, b, c through τ. For this we notice that subtraction of Einstein equations (3)-(4) leads to the equation

$$\frac{\ddot a}{a} - \frac{\ddot b}{b} + \frac{\dot a \dot c}{ac} - \frac{\dot b \dot c}{bc} = \frac{d}{dt}\Big(\frac{\dot a}{a} - \frac{\dot b}{b}\Big) + \Big(\frac{\dot a}{a} - \frac{\dot b}{b}\Big)\Big(\frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c}\Big) = 0. \tag{25}$$

Equation (25) possesses the solution

$$\frac{a}{b} = D_1 exp\Big(X_1 \int \frac{dt}{\tau}\Big), \quad D_1 = const, \quad X_1 = const. \tag{26}$$

Subtracting equations (3)-(5) and (4)-(5) one finds the equations similar to (25), having solutions

$$\frac{a}{c} = D_2 exp\Big(X_2 \int \frac{dt}{\tau}\Big), \quad \frac{b}{c} = D_3 exp\Big(X_3 \int \frac{dt}{\tau}\Big), \tag{27}$$

where D_2, D_3, X_2, X_3 are integration constants. There is a functional dependence between the constants $D_1, D_2, D_3, X_1, X_2, X_3$:

$$D_2 = D_1 D_3, \qquad X_2 = X_1 + X_3. \tag{28}$$

Using the equations (26), (27) and (28), we rewrite $a(t), b(t), c(t)$ in the explicit form:

$$\begin{aligned} a(t) &= (D_1^2 D_3)^{\frac{1}{3}} \tau^{\frac{1}{3}} exp\Big[\frac{2X_1 + X_3}{3} \int_{t_0}^{t} \tau^{-1} dt'\Big] \\ b(t) &= (D_1^{-1} D_3)^{\frac{1}{3}} \tau^{\frac{1}{3}} exp\Big[-\frac{X_1 - X_3}{3} \int_{t_0}^{t} \tau^{-1} dt'\Big] \\ c(t) &= (D_1 D_3^2)^{-\frac{1}{3}} \tau^{\frac{1}{3}} exp\Big[-\frac{X_1 + 2X_3}{3} \int_{t_0}^{t} \tau^{-1} dt'\Big] \end{aligned} \tag{29}$$

where t_0 is the initial time.

Thus the previous system of Einstein equations and interacting spinor and scalar field ones is completely integrated. In this process of integration only first three of the complete system of Einstein equations have been used. General solutions to these three second order equations have been obtained. The solutions contain six arbitrary constants: D_1, D_3, X_1, X_3 and two others C_1 and t_0, that were obtained while solving equation (22). Equation (6) is the consequence of first three of Einstein equations. To verify the correctness of obtained solutions, it is necessary to put $a(t), b(t)$ and $c(t)$ in (6). It should lead either to identity or to some additional constraint between the constants. Putting $a(t), b(t), c(t)$ from (29) in (6) one can get the following equality:

$$\frac{1}{3\tau}\Big[3\ddot\tau - 2\frac{\dot\tau^2}{\tau} + \frac{2}{3\tau}\Big(X_1^2 + X_1X_3 + X_3^2\Big)\Big] = -\kappa\Big(T_0^0 - \frac{1}{2}T\Big), \tag{30}$$

that guaranties the correctness of obtained solutions.

To get the constant C_1 in (24) one can use the equation (30). Inserting $\ddot{\tau}$ from (22), $\dot{\tau}^2$ from (23) and

$$T_0^0 - \frac{1}{2}T = \frac{i}{4}N + \frac{1}{2}\frac{C^2}{\tau^2}P + R = \frac{mC_0}{2\tau} + \frac{C^2}{\tau^2}P + \frac{3C_0C^2}{4\tau^3}P',$$

one deduces the identity, if

$$C_1 = \frac{1}{9\kappa}\Bigl(X_1^2 + X_1X_3 + X_3^2\Bigr), \tag{31}$$

which means that the constant C_1 is a positive one.

We will first study the solution to the system of field equations with minimal coupling when the direct interaction between the spinor and scalar fields remains absent, i.e. in the Lagrangian (1) $\Phi(S) \equiv 1$. The reason to get the solution to the self-consistent system of equations for the fields with minimal coupling is the necessity of comparing this solution with that for the system of equations for the interacting spinor, scalar and gravitational fields that permits to clarify the role of interaction terms in the evolution of the cosmological model in question.

In this case the components of the energy-momentum tensor look:

$$\begin{aligned} T_0^0 &= \frac{mC_0}{\tau} + \frac{C^2}{2\tau^2}, \quad T_1^1 = T_2^2 = T_3^3 = -\frac{C^2}{2\tau^2}, \\ T &= T_\alpha^\alpha = \frac{mC_0}{\tau} - \frac{C^2}{\tau^2}, \quad T_1^1 + T_2^2 + T_3^3 - \frac{3}{2}T = -\frac{3}{2}\frac{mC_0}{\tau}. \end{aligned} \tag{32}$$

Note that as the energy density T_0^0 should be a quantity positively defined, the equation (32) leads to $C_0 > 0$. The inequality $C_0 > 0$ will also be preserved for the system with direct interaction between the fields as in this case the correspondence principle should be fulfilled: for $\lambda = 0$ the field system with direct interaction turns into that with minimal coupling.

Taking into account (32) equation (22) writes

$$\ddot{\tau} = \frac{3}{2}\kappa m C_0, \tag{33}$$

with the solution

$$\tau(t) = \frac{3}{4}\kappa m C_0 t^2 + \tau_1 t + \tau_2, \quad \tau_1, \tau_2 = const. \tag{34}$$

Putting $\tau(t)$ from (34) into (18) and (29) one gets the explicit expressions for the components of spinor field functions $V_\rho(t)$ and metric functions $a(t), b(t), c(t)$:

$$V_r(t) = \frac{C_r}{\sqrt{\tau}}e^{-imt}, \quad V_l(t) = \frac{C_l}{\sqrt{\tau}}e^{imt}, \tag{35}$$

$$\begin{aligned} a(t) &= (D_1^2D_3)^{\frac{1}{3}}\tau^{\frac{1}{3}}Z^{\frac{2X_1+X_3}{3}}, \\ b(t) &= (D_1^{-1}D_3)^{\frac{1}{3}}\tau^{\frac{1}{3}}Z^{-\frac{X_1-X_3}{3}}, \\ c(t) &= (D_1D_3^2)^{-\frac{1}{3}}\tau^{\frac{1}{3}}Z^{-\frac{X_1+2X_3}{3}}, \end{aligned} \tag{36}$$

where

$$Z = \left(\frac{t-t_1}{t-t_2}\right)^\sigma, \quad \sigma = \frac{4}{3\kappa m C_0 (t_1 - t_2)}, \tag{37}$$

and $t_{1,2} = -\frac{2\tau_1}{3\kappa m C_0} \pm \frac{2}{3\kappa m C_0}\sqrt{\tau_1^2 - 3\kappa m C_0 \tau_2}$ are the roots of the quadratic polinomial in the right-hand side of (34). If the roots are real, i.e. if

$$\tau_1^2 - 3\kappa m C_0 \tau_2 \geq 0, \tag{38}$$

the solution (34) is singular one, while in the opposite case it is not. Putting (34) into (30) one deduces the following relation between the constants:

$$\tau_1^2 - 3\kappa m C_0 \tau_2 = \frac{3}{2}\kappa C^2 + \frac{1}{3}\Big(X_1^2 + X_1 X_3 + X_3^2\Big). \tag{39}$$

As the right-hand side of the equation (39) is positive, the quadratic trinomial in (34) possesses real roots and the solution obtained is singular one at initial time $t = t_1$, whereas $t_1 > t_2, t_1 \leq t \leq \infty$.

Let us study the solution (34)-(36) at $t \to \infty$. Hence we have: $\tau(t) \approx \frac{3}{4}\kappa m C_0 t^2$, and $a(t) \sim b(t) \sim c(t) \sim t^{2/3}$, that leads to the conclusion about the asymptotical isotropization of the expansion process for the initially anisotropic Bianchi type-I space-time.

Thus the solution to the self-consistent system of equations for the spinor, scalar and gravitational fields is the singular one at the initial time. In the initial state of evolution of the field system the expansion process of space-time is anisotropic, but at $t \to \infty$ there happens isotropization of the expansion process.

To investigate the system of spinor and scalar field equations with direct interaction we will consider the partial case for choosing $P(S)$:

$$P(S) = 1 + \lambda S^n = 1 + \lambda \frac{C_0^n}{\tau^n}, \tag{40}$$

where λ is the interaction parameter, n is some arbitrary constant. Inserting (40) into (24) one obtains

$$\int \frac{d\tau}{\sqrt{mC_0\tau + \lambda C^2 C_0^n / 2\tau^n + C_2^2}} = \sqrt{3\kappa} t, \tag{41}$$

where $C_2^2 = C^2/2 + C_1$; in (24) t_0 has been taken zero, as it only gives the shift of the initial time.

Let us study different cases of choosing λ and n. I. $\lambda > 0, \quad n > 0$. In this case (41) leads to the following behavior of $\tau(t)$:

$$at \quad t \to \infty \quad \tau(t) \approx \frac{3}{4}\kappa m C_0 t^2 \to \infty, \tag{42}$$

$$at \quad t \to 0 \quad \tau(t) \approx \left[\Big(\frac{n}{2} + 1\Big)\sqrt{\frac{3\kappa\lambda C_0^n C^2}{2}} t\right]^{\frac{1}{n/2+1}} \to 0. \tag{43}$$

Note that (42) coincides with (34) at $t \to \infty$. It leads to the fact that in the case considered, the asymptotical isotropization of the expansion process of initially anisotropic

Bianchi type-I space-time takes place without the influence of scalar field. In this case the initial state is singular: $\tau(0) = 0$.

Thus, the evolution of the interacting fields system at $\lambda > 0$ and $n > 0$ is qualitatively the same as that of the system with minimal coupling.

II. $\lambda = -\sigma^2 < 0, \quad n > 0$. The equation (41) takes the form:

$$\int \frac{d\tau}{\sqrt{mC_0\tau - \sigma^2 C^2 C_0^n / 2\tau^n + C_2^2}} = \sqrt{3\kappa}t. \tag{44}$$

From (44) follows:

$$at \quad t \to \infty \quad \tau(t) \approx \frac{3}{4}\kappa m C_0 t^2 \to \infty,$$

i.e. as well as in the previous case the asymptotical isotropization of the expansion process of initially anisotropic Bianchi type-I space-time takes place. But $\tau = 0$ cannot be reached as in this case the denominator of the integrand in (44) becomes imaginary at $\tau \to 0$. There exists the minimum value $\tau_{min} = \tau_0 > 0$, which is defined from the equation

$$mC_0\tau_0^{n+1} + C_2^2\tau_0^n - \frac{\sigma^2 C^2 C_0^n}{2} = 0.$$

It means that for $\lambda < 0$ and $n > 0$ there exist regular solutions to the previous system of equations. The absence of the initial singularity in the considered cosmological solution appears to be consistent with the violation for $\lambda < 0$, of the dominant energy condition in the Hawking-Penrose theorem [1].

III. $\lambda > 0, \quad n = -k^2 < 0$. In this case the equation (41) takes the form:

$$\int \frac{d\tau}{\sqrt{mC_0\tau + \lambda C^2 \tau^{k^2} / 2C_0^{k^2} + C_2^2}} = \sqrt{3\kappa}t. \tag{45}$$

Let us study concrete solutions for some values of k^2.

a) $k^2 = 1$. Then from (45) one gets:

$$\tau(t) = \frac{3}{4}MC_0\kappa t^2 - \frac{C_2^2}{MC_0}, \quad M = m + \frac{\lambda C^2}{2C_0^2}. \tag{46}$$

The solution (46) is singular one at initial time $t_0 = \frac{2C_2}{\sqrt{3\kappa}MC_0}$ and asymptotically isotropic.

b) $k^2 = 2$. The equation (45) writes

$$\int \frac{d\tau}{\sqrt{mC_0\tau + \lambda C^2\tau^2/2C_0^2 + C_2^2}} = \sqrt{3\kappa}t. \tag{47}$$

Integration of (47) leads to

$$\tau(t) = \frac{C_0^2}{\lambda C^2}\left[\triangle sh\left(\frac{\sqrt{3\kappa\lambda}C}{\sqrt{2}C_0}t\right) - mC_0\right], \tag{48}$$

$$\triangle = \sqrt{2\lambda C^2 C_2^2 / C_0^2 - m^2 C_0^2}.$$

From (48) one gets: $\tau(t_0) = 0$, where t_0 is defined from the equation:

$$\triangle sh\left(\frac{\sqrt{3\kappa\lambda}C}{\sqrt{2}C_0}t_0\right) - mC_0 = 0, \tag{49}$$

i.e. the solution (48) is singular at initial time $t = t_0$.

At $t \to \infty$

$$\tau(t) \approx \frac{C_0^2}{\lambda C^2}\triangle exp\left(\frac{\sqrt{3\kappa\lambda}C}{\sqrt{2}C_0}t\right). \tag{50}$$

The solution (48) describes initially (i.e. at t_0) singular and asymptotically (i.e. at $t \to \infty$) isotropic Bianchi Type-I cosmological model. Note that in this case the transition to the isotropic regime happens exponentially.

IV. $\lambda = -\sigma^2 < 0, \quad n = -k^2 < 0$. In this case the equation (41) takes the form:

$$\int \frac{d\tau}{\sqrt{mC_0\tau - \sigma^2 C^2 \tau^{k^2}/2C_0^{k^2} + C_2^2}} = \sqrt{3\kappa}t. \tag{51}$$

Let us consider concrete solutions for some values of k^2 as in III.

a) $k^2 = \frac{1}{2}$. In this case one gets from (51):

$$\frac{2}{\sqrt{mC_0}}\left(\sqrt{\tau} + \sqrt{\tau_1} ln \mid \sqrt{\tau} - \sqrt{\tau_1} \mid\right) = \sqrt{3\kappa}t, \quad \sqrt{\tau_1} = \frac{\sigma^2 C^2}{4mC_0^{3/2}}, \tag{52}$$

which leads to

$$at \quad t \to \infty \quad \tau(t) \approx \frac{3\kappa m C_0}{4}t^2 \to \infty, \tag{53}$$

$$at \quad t \to -\infty \quad \tau(t) \to \tau_1 = \left(\frac{\sigma^2 C^2}{4mC_0^{3/2}}\right)^2. \tag{54}$$

From (53) and (54) one comes to the conclusion that the solution (52) is initially (i.e. at $t_0 = -\infty$) regular one and at $t \to \infty$ asymptotically isotropic.

b) $k^2 = 1$. In this case (51) writes:

$$\int \frac{d\tau}{\sqrt{\left(m - \sigma^2 C^2/2C_0^2\right)C_0\tau + C_2^2}} = \sqrt{3\kappa}t. \tag{55}$$

If in (55) $m - \sigma^2 C^2/2C_0^2 > 0$, then the solution coincides with (46), where $M = m - \sigma^2 C^2/2C_0^2$. At $m - \sigma^2 C^2/2C_0^2 = -T^2 < 0$ from (55) one gets

$$\tau(t) = \frac{C_2^2}{T^2 C_0} - \frac{3\kappa T^2 C_0}{4}t^2. \tag{56}$$

In this case $\tau(t)$ possesses

$$maximum \quad at \quad t = 0, \quad i.e. \quad \tau(0) = \frac{C_2^2}{T^2 C_0},$$

$$and \quad minimum \quad at \quad t_{1,2} = \mp\frac{2C_2}{\sqrt{3\kappa}T^2 C_0} \quad i.e. \quad \tau(t_{1,2}) = 0. \tag{57}$$

The solution obtained describes the cosmological model, which begins to expand at t_1, acquires its maximum at $t = 0$ and then collapses into a point at t_2.

c) $k^2 = 2$. From (45) one gets:

$$\int \frac{d\tau}{\sqrt{mC_0\tau - \sigma^2 C^2 \tau^2/2C_0^2 + C_2^2}} = \sqrt{3\kappa} t. \tag{58}$$

Integrating (58), for $\tau(t)$ one gets the following expression:

$$\tau(t) = \frac{C_0^2}{\sigma^2 C^2}\Big[mC_0 + \triangle sin\Big(\frac{\sigma C\sqrt{3\kappa}t}{\sqrt{2}C_0}\Big)\Big], \tag{59}$$

where $\triangle = \Big(m^2 C_0^2 + \frac{2\sigma^2 C^2 C_2^2}{C_0^2}\Big)^{1/2}$. From (59) follows that $\tau(t_0) = 0$, where

$$t_0 = -\frac{\sqrt{2}C_0}{\sqrt{3\kappa}\sigma C} arcsin\Big(\frac{mC_0}{\triangle}\Big), \tag{60}$$

then acquires maximum

$$\tau(t_{max}) = \frac{C_0^2}{\sigma^2 C^2}\Big(mC_0 + \triangle\Big),$$

where

$$t_{max} = \frac{\pi\sqrt{2}C_0}{2\sqrt{3\kappa}\sigma C},$$

and further at $t = t_1$ again turns to zero: $\tau(t_1) = 0$, where

$$t_1 = \pi + \frac{\sqrt{2}C_0}{\sqrt{3\kappa}\sigma C} arcsin\Big(\frac{mC_0}{\triangle}\Big).$$

Thus the solution (59) describes the cosmological model, which begins to expand at t_0, acquires its maximum at t_{max} and then collapses into a point at t_1.

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