GENERALIZED EQUATIONS OF ELECTRODYNAMICS OF CONTINUOUS MEDIA[1]

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Development of quantum engineering put forward new theoretical problems. Behavior of a single mesoscopic cell (device) we may usually describe by equations of quantum mechanics. However if experimentators gather hundreds of thousands of similar cells there arises some artificial medium that one already needs to describe by means of electromagnetic equations. In the present work it is demonstrated that the inherent primacy of vector potential in quantum systems leads to a generalization of the equations of electromagnetism by introducing in them toroid polarizations. We mention some of their applications.

Key words: Toroid multipole moments, Toroidomagnetics, Electromagnetotoroidics, Toroidomagnetostatics, Magnetoelectronics

[1] "The present status of the Quantum Theory of Light", (S. Jeffers et al. eds), Kluwer Academic Publishers, pp. 141-150 (1997).

This report is devoted to **Prof. Jean-Pierre Vigier** who has made very valuable contributions in the development of Quantum Mechanics and Electrodynamics. It is well-known that these two disciplines overlap within the scope of atomic physics. Electrodynamics of continuous media concerns much more intricate problems in both field and matter aspects. We show that usual equations of electromagnetic media are incomplete even in their fundamental representation and correct this oversight. It should be emphasized that we are not dealing with innovations based on additional (even plausible) hypotheses but with an inevitable modification imperatively following merely from the facts of our three-dimensional life. Our report consists of two parts. The first one can be considered as some formal deductions. Its starting point is the demonstration of existence of the third family of multipole moments – the *toroid* one in multipole expansion of electromagnetic current (for their first strict introduction see ref. [1]). It was just the cause that made it necessary to modify the equations of electrodynamics of continuous media. Simply speaking, in addition to electrical polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ one has to introduce toroid polarization $\mathbf{T}$ in the (vector!) equations. The impact of this operation is not trivial at all. The matter is that the toroid moments are the multipole sources of free–field potentials [2, 3], that are responsible in particular for effects like the Aharonov–Bohm one. In fact, we know that only quantum particles can serve as a detector of this potential. Thus, there arises a series of principle questions. For example, questions relating to the transition from quantum mechanical description of electromagnetic phenomena to the description with the help of classical equations. This concerns the profound physical problems that will not be discussed here (see, e.g. [4]). Now we will give a high-light of the history of the discovery of toroid moments and how they were associated with experiments to study the Aharonov–Bohm effect. Recently A. Tonomura and others [5] have observed the interference of electrons on a shielded ferromagnetic ring of mesoscopic size. Distribution of vector-potential created by the source mentioned needed computation. The most detailed calculation of this distribution was done by G. Afanas'ev [6]. Shortly after that it was noticed that the toroid dipole moment plays the role of a point–like source of this kind of distribution [2]. Let us explain what this moment is? From the geometrical point of view its model is the poloidal current on a torus [Fig. 1]. Macroscopically, its model is created by the usual toroid coil with an even number of windings. There is a hydrodynamical analog of this construction – Hill's vortex [Fig. 2].

It is easy to demonstrate how in the system of three particles one can emphasize all three dipole moments [7]. Suppose that a steady system consists of the sun (S), the earth (E) and the moon (M). Suppose the earth and the moon are oppositely charged and the sun is neutral [Fig. 3a]. Then, in each given instant it may be convenient to describe the subsystem E–M by an electric dipole moment $\mathbf{d}$. If the intrinsic angular velocity $\mathbf{\Omega}_d$ in the E-M subsystem is high in comparison with its external rotation around S, $\mathbf{\Omega}_m << \mathbf{\Omega}_d$, we may observe the magnetic dipole $\mathbf{m}$ [Fig. 3b]. If $\mathbf{\Omega}_m$ increases, we have to take into account the toroid properties of the system, i.e., the toroid dipole $\boldsymbol{\tau}$. **Remark:** As far as in the atoms and nuclei their magnetic fluxes are mainly confined inside these many–body systems, they can possess great toroid moments. Moreover, it is not difficult to show that $[L_z, \hat{\boldsymbol{\tau}}_z] = 0$ and, for example, in the external nonuniform and/or alternating fields we may observe the effect of atomic spectral line splitting additional to the Stark's and Zeeman's ones [1]. How does the dipole moment $\boldsymbol{\tau}$ arise in electromagnetic current distribution? From the formal point of view 4-currents possess 4 scalar components and each of them can be expanded in multipole series. Implying one constrain, the current conservation condition, we should obtain three families of moments and in each of them will have its proper dipole. (Their definitions are given in Table – 1). A renowned soviet physicist Ya. B. Zeldovich [8] was the first to notice that the toroid coil was impossible to identify with any multipole moments, starting with quite different considerations. He assumed that as the third dipole one could take the classical analog of P-odd form-factor of spin 1/2 particle, named the anapole. However, further it was proved that this kind of assumption was not completely correct. For example, the anapole cannot radiate at all while the toroid coil can. The matter is that the anapole is some composition of electric dipole and actual toroid dipole giving destructive interference of their radiation. Creation of a complete theory of multipole expansion appeared to be a very intricate problem. The first correct and vast article [1] on this topic was published only in 1974 and ref. [9] can be considered as the last one. In the theory of continuous media, there appear possibilities of introducing more families of multipole moments [9], e.g. for electric–dipole medium $\boldsymbol{\tau}_e$ [10]. Four dipole moments $\mathbf{d}, \mathbf{m}, \boldsymbol{\tau}$ and $\boldsymbol{\tau}^e$ manifest all possible combinations of properties at inversions of space and time and form a complete vector basis of order parameters for describing crystalline substances [11]. By this time, possibilities and demands of physicists and technologists have been also increased. For example, now-a-days system of thousands of mesoscopic ferrite rings is being produced, studied and applied. However, the appropriate experimental and theoretical results in consideration don't coincide yet [12, 13]. What kind of equations can describe the properties and response of such magnetic medium? Magnetic field in this medium is confined inside the rings unlike outside of it there is only the distribution of free–field vector-potential. The first order equations for $\mathbf{E}$ and $\mathbf{B}$, which are called the Maxwellian, are not obviously sufficient for this purpose. We offer a new two potential formulation (see also [14]).

1. Static dipole moment of toroid coil and a free-field potential created by it We begin with a static problem. Let us first find the distribution of the vector potential $\mathbf{A}$ produced by a "point–like" poloidal current $\mathbf{I}$. The toroid dipole moment of the toroidal coil is $\boldsymbol{\tau} = \mathbf{I}V$, where V is the volume of the coil (torus). In the (quasi)static case, the basic equation (with the gauge condition div$\mathbf{A} = 0$, valid outside the source) has the form

$$\mathrm{curl\ curl}\,\mathbf{A} = \mathrm{curl\ curl}\,\boldsymbol{\tau}\,\delta(\mathbf{r}). \tag{1}$$

Its solution is a convolution of two distributions, the Green function and the δ- function, and is to be determined on a suitable test vector function. Thus we may get [2]

$$\begin{aligned}\mathbf{A} &= \mathrm{curl\ curl}\,\boldsymbol{\tau} r^{-1} = \boldsymbol{\tau}\triangle(1/r) + \boldsymbol{\tau}\cdot\nabla\nabla(1/r) = \\ &= \frac{3\mathbf{r}\mathbf{r}\cdot\boldsymbol{\tau} - r^2\boldsymbol{\tau}}{r^5} + \boldsymbol{\tau}\delta(\mathbf{r}).\end{aligned} \tag{2}$$

We see that the toroid dipole moment $\boldsymbol{\tau}$ produces the potential distribution $\mathbf{A}$, just like $\mathbf{d}$ produces the electric field $\mathbf{E}$ and $\mathbf{m}$ the magnetic induction $\mathbf{B}$. Therefore, for media

$$\mathbf{D} = \mathbf{E} + \mathbf{P} \quad \text{and} \quad \mathbf{B} = \mathbf{H} + \mathbf{M}. \tag{3}$$

What about $\mathbf{A}$ and $\boldsymbol{\tau}$?

2. Generalized equations of electromagnetism Let us note that as early as 1977, V. Dubovik with his collaborators showed that the crystal media in general can hardly be described without introducing polar and axial toroid polarizations [10]. Even then, the question of generalizing the fundamental equations of electrodynamics of continuous media came to the light. First, they were presented at the seminar of LTPh, JINR in January 1991 and published in 1994 [14]. In this paper, we will present the inductive and deductive foundations of these equations and some of their consequences. Let us write the magnetostatic equations in their two equivalent forms (see for example [15])

$$\mathrm{curl}\,\mathbf{H} \;=\; \mathbf{j}, \tag{4a}$$

$$\mathrm{div}\,\mathbf{H} \;=\; -\,\mathrm{div}\,\mathbf{M}, \quad \mathbf{M} \;=\; \mathbf{M}(\mathbf{H}), \tag{4b}$$

$$\mathrm{curl}\,\mathbf{H} \,+\, \mathrm{curl}\,\mathbf{M} \;=\; \mathrm{curl}\,\mathbf{B} \;=\; \mathbf{j}^{\mathrm{free}} + \mathrm{curl}\,\mathbf{M}, \tag{4c}$$

$$\mathrm{div}\,\mathbf{B} \;=\; 0, \quad \mathbf{M} \;=\; \mathbf{M}(\mathbf{B}). \tag{4d}$$

It is easy to see that without introducing the vector-potential we cannot describe an arbitrary magnetic medium, e.g., consisting of closed chains of magnet, i.e., dipoles (e.g., in the form of a ring [Table 2]). Really, in this case the macroscopic pattern of such a medium has $\mathbf{M} \equiv 0$ and in the absence of free currents we obtain $\mathbf{B} \equiv \mathbf{H}$. Then, the magnetostatic equations are trivialized:

$$\begin{aligned} \mathrm{curl}\,\mathbf{B} &= 0, \qquad \mathrm{div}\,\mathbf{B} = 0, \\ \mathrm{curl}\,\mathbf{H} &= 0, \qquad \mathrm{div}\,\mathbf{H} = 0, \end{aligned} \tag{4}$$

from where it seems that we should conclude, according to the Helmholtz theorem, that $\mathbf{B} \equiv 0$ and $\mathbf{H} \equiv 0$ all over the space. But it is not correct. The fact here is that $\mathbf{M} \equiv 0$ is taken on average, but each physical volume, occupied by a closed chain, becomes topologically non-trivial one! So we rewrite the magnetostatic equation (4c) through $\mathbf{a}$ (an analog of $\mathbf{H}$) *at the same time* adding the contribution of toroid polarization to the left- as well as right-hand sides, like in the transition from (4a) to (4c):

$$\mathrm{curl}\ \ \mathrm{curl}\,\mathbf{a} + \mathrm{curl}\ \ \mathrm{curl}\,\mathbf{T} = \mathbf{j}^{\mathrm{free}} + \mathrm{curl}\,\mathbf{M} + \mathrm{curl}\ \ \mathrm{curl}\,\mathbf{T}, \tag{5}$$

and at all points of space we introduce

$$\boldsymbol{\alpha} := \mathbf{a} \,+\, \mathbf{T}. \tag{6}$$

In the absence of free charges, we may put for $\boldsymbol{\alpha}$ (an analog of $\mathbf{B}$) that $\mathrm{div}\,\boldsymbol{\alpha} \;=\; 0$. Consequently, the fundamental equations of toroidomagnetostatics can be written as

$$\mathrm{curl}\ \ \mathrm{curl}\,\boldsymbol{\alpha} = \mathbf{j}^{\mathrm{free}} + \mathrm{curl}\,\mathbf{M} + \mathrm{curl}\ \ \mathrm{curl}\,\mathbf{T}, \tag{7}$$

$$\mathrm{div}\,\boldsymbol{\alpha} = 0, \tag{8}$$

with the relation (6), that holds all over the space. The latter system is the analog of the equations for $\mathbf{B}$. The inverse reduction of this system may be considered as an analog of equations (4a) and (4b) for $\mathbf{H}$, naturally appears as (4a) and (4b)

$$\mathrm{curl}\ \ \mathrm{curl}\,\mathbf{a} = \mathbf{j}^{\mathrm{free}} + \mathrm{curl}\,\mathbf{M}, \tag{9}$$

$$\mathrm{div}\,\mathbf{a} = -\mathrm{div}\,\mathbf{T}. \tag{10}$$

Employing the Helmholtz theorem, we obtain the solution to the latest system in the form:

$$\mathbf{a}(\mathbf{r}) \;=\; \int \frac{\mathbf{j}^{\mathrm{free}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' + + \int \frac{\mathrm{curl}\,\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' \,+\, \nabla \int \frac{\mathrm{div}\,\mathbf{T}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r'. \tag{11}$$

The first two terms give the contribution, which is usually denoted as magnetic vector potential $\mathbf{A}(\mathbf{r})$; in our notation, the *total* vector-potential is $\boldsymbol{\alpha} = \mathbf{a} + \mathbf{T}$. Obviously, the definition of the magnetic field is also changed due to toroid polarization (which can be nonhomogeneous for the given concrete medium, at least in the form of the surface effect [11]) as follows

$$\boldsymbol{\beta} := \operatorname{curl}\boldsymbol{\alpha} = \operatorname{curl}\int \frac{\mathbf{j}^{\text{free}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' + \operatorname{curl}\int \frac{\operatorname{curl}\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' + \operatorname{curl}\mathbf{T}(\mathbf{r}). \tag{12}$$

As one can see, the first two terms gives the usual definition of magnetic field (according to the old terminology, the magnetic induction) in a medium, where as the magnetic field in the much of crystals, (e.g., without inversion center of a cell [3]) or on the surface of a crystal, studied by the experimentallists (e.g. by means of magneto–optical devices), will be also contributed by the third term. Remark, naturally, the toroidomagnetostatic equation turns into the wave equation if one adds to curl curl $\boldsymbol{\alpha}$ the term $\ddot{\boldsymbol{\alpha}}$

$$\Box\,\boldsymbol{\alpha} = \operatorname{curl}\mathbf{M} + \operatorname{curl}\ \operatorname{curl}\mathbf{G}. \tag{13}$$

Immediate generalization of equations of electromagnetism may be schematically made as follows. If in a given medium there are no free charges and currents, it can be described by the usual transverse equation:

$$\operatorname{curl}\mathbf{D} + \dot{\mathbf{B}} = \operatorname{curl}\mathbf{P}, \quad \operatorname{div}\mathbf{D} = 0, \tag{14}$$

$$\operatorname{curl}\mathbf{B} - \dot{\mathbf{D}} = \operatorname{curl}\mathbf{M}, \quad \operatorname{div}\mathbf{B} = 0. \tag{15}$$

We may now transit to the 2–potential formulation through $\boldsymbol{\alpha}^m$ and $\boldsymbol{\alpha}^e$ and introduce electric and magnetic toroid polarizations $\mathbf{T}^e$ and $\mathbf{T}^m$ through substitution [14]:

$$\mathbf{D} \Longrightarrow -\dot{\boldsymbol{\alpha}}^m + \operatorname{curl}\boldsymbol{\alpha}^e, \quad \operatorname{curl}\mathbf{P} \Longrightarrow \operatorname{curl}\mathbf{P} + \operatorname{curl}\operatorname{curl}\mathbf{T}^e, \tag{16}$$

$$\mathbf{B} \Longrightarrow \dot{\boldsymbol{\alpha}}^e + \operatorname{curl}\boldsymbol{\alpha}^m, \quad \operatorname{curl}\mathbf{M} \Longrightarrow \operatorname{curl}\mathbf{M} + \operatorname{curl}\operatorname{curl}\mathbf{T}^m. \tag{17}$$

Then, we obtain

$$\ddot{\boldsymbol{\alpha}}^e + \operatorname{curl}\ \operatorname{curl}\boldsymbol{\alpha}^e = \operatorname{curl}\mathbf{P} + \operatorname{curl}\operatorname{curl}\mathbf{T}^e, \tag{18}$$

$$\ddot{\boldsymbol{\alpha}}^m + \operatorname{curl}\ \operatorname{curl}\boldsymbol{\alpha}^m = \operatorname{curl}\mathbf{M} + \operatorname{curl}\operatorname{curl}\mathbf{T}^m. \tag{19}$$

If we choose a gauge condition $\operatorname{div}\boldsymbol{\alpha}^{e,m} = 0$, we may again obtain the form (13).

conclusion

"$\cdots$ it is impossible to introduce
electrodynamics of "matter in general"
- from the book by Russian academician **E. A. Turov (1983)**

It should be noticed that we do not consider contributions of high multipole moments in the Maxwell equations. Here, we develop only the macroscopic description of electromagnetotoroidic dipole media. There arose a large field of activity to model the material equations of concrete media. We did not consider here the problem of alignment of microscopic toroid moments by crystalline fields. Ideal static toroid moments do not interact with each other at all. However, toroidization can appear due to dynamical effects (see e.g. [16]). Among the latest machinery we point out the articles [17, 18] that directly precede applications of toroid moments in the area of high technologies.

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Table 1. The models and the definitions of three kinds of current dipole moments: a bit of the linear current, a ring with the circular current, the toroidal coil.

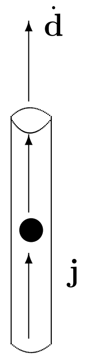
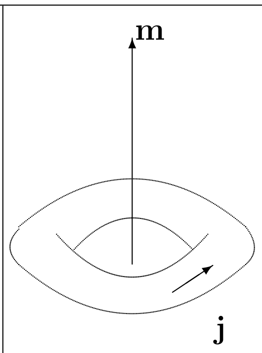
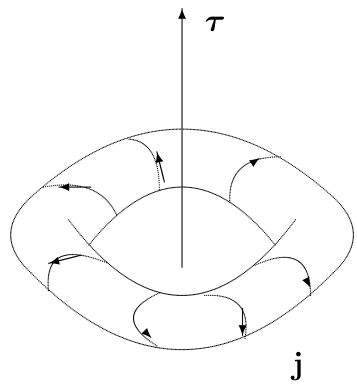
		
$\dot{\mathbf{d}} = \int \mathbf{j}\, dV$	$\mathbf{m} = \frac{1}{2} \int [\mathbf{r} \times \mathbf{j}] dV$	$\boldsymbol{\tau} = \frac{1}{10} \int [\mathbf{r}(\mathbf{r} \cdot \mathbf{j}) - 2r^2 \mathbf{j}] dV$
$\dot{\mathbf{d}} = I\mathbf{l}$	$\mathbf{m} = \frac{1}{2} I\mathbf{S}$	$\boldsymbol{\tau} \approx \Phi\mathbf{S}, \lvert\boldsymbol{\tau}\rvert = IV_{\text{torus}}$

Table 2. Basic properties of the dipole moments under the spatial and temporal inversions and their interactions with external fields.

	$\mathbf{P}(\mathbf{r} \rightarrow -\mathbf{r})$	$T(t \rightarrow -t)$	
$\mathbf{d}$	$-$	$+$	$\mathbf{d} \cdot \mathbf{E}$
$\mathbf{m}$	$+$	$-$	$\mathbf{m} \cdot \mathbf{B}$
$\boldsymbol{\tau}^e$ and $\boldsymbol{\tau}^m$	$-$	$-$	$\boldsymbol{\tau} \cdot \dot{\mathbf{D}}$ or $\boldsymbol{\tau} \text{curl}\, \mathbf{B}$
$\boldsymbol{\tau}^e$	$-$	$-$	$\boldsymbol{\tau}^e \cdot \dot{\mathbf{B}}$ or $\boldsymbol{\tau}^e \text{curl}\, \mathbf{E}$

Figure 1. Poloidal current on the torus determines the toroid dipole moment. The simplest model of this is an ordinary toroidal coil with even number of winding.

Figure 2. Poloidal lines of currents in a simply connected volume, the sphere ($\theta = const.$, where θ is the polar angle of spherical coordinate system). It is a model of the Hill's vortex.

Figure 3a. Microscopically we see two separate charges and at each instant the electric dipole moment $\mathbf{d} = q\,\mathbf{r}_q$.

Figure 3b. If $\Omega_m \to 0$ we "see" the magnetic dipole (average on the motion of the moon) $|\mathbf{m}| = \pi\, I_d\, r_d^2$, where $I_d = -q\,\Omega_d/2\pi$.

Figure 3c. If $\Omega_d \gg \Omega_m$ we "see" global macroscopic the toroid dipole moment (average on the motion both of the moon and the earth) of total system equals $|\boldsymbol{\tau}| = |\mathbf{m}|\mathbf{r_m}$.

Figure 4. Geometrical illustration of polar $\mathbf{T}^m = \frac{1}{2}\sum_i [\mathbf{r}^{(i)} \times \mathbf{m}^{(i)}]$ and axial $\mathbf{T}^e = \frac{1}{2}\sum_i [\mathbf{r}^{(i)} \times \mathbf{d}^{(i)}]$ toroid vectors in magnetic and electric dipole media.

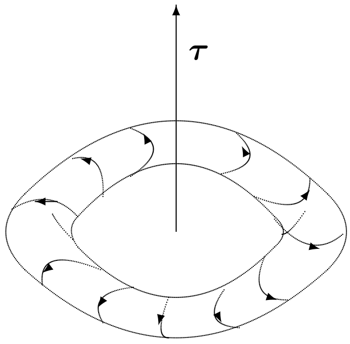


Figure 1

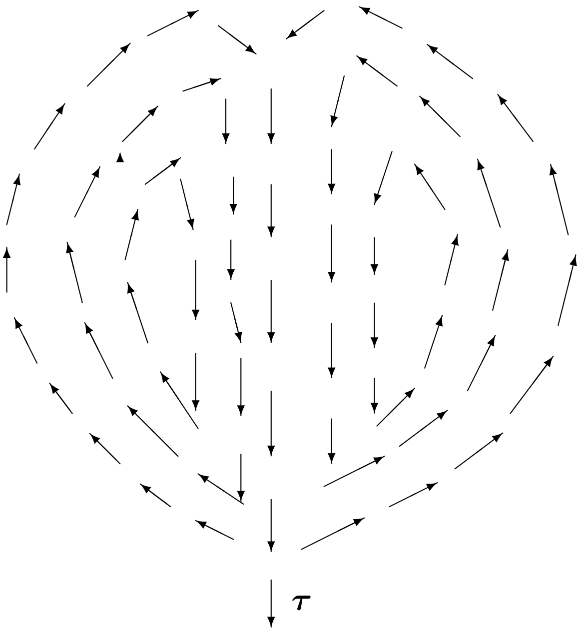


Figure 2

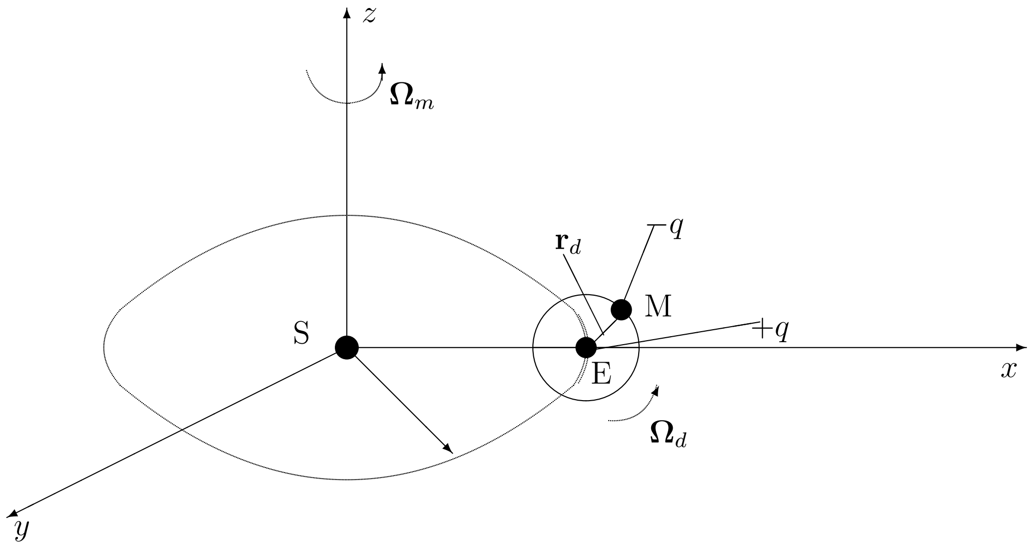


Figure 3a

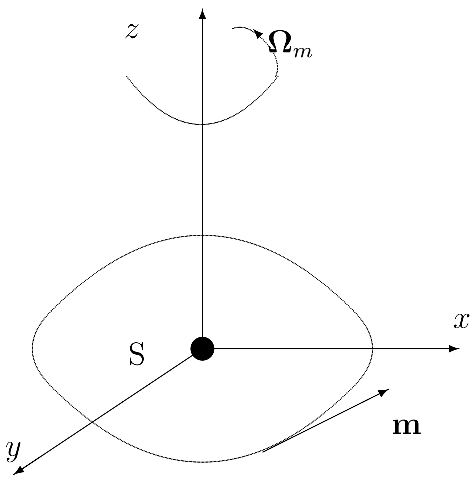


Figure 3b

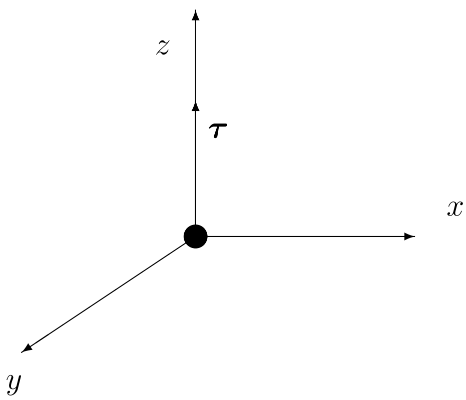
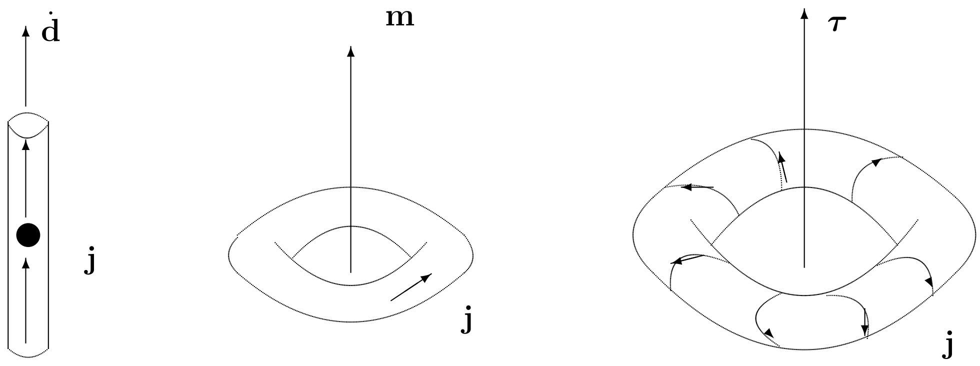


Figure 3c

Figure 4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