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Generating function method for constructing new iterations

T. Zhanlav[a], O. Chuluunbaatar[a,b], V. Ulziibayar[a,c,*]

[a] Institute of Mathematics, National University of Mongolia, Mongolia
[b] Joint Institute for Nuclear Research, Dubna 141980, Moscow region, Russia
[c] School of Applied Sciences, Mongolian University of Science and Technology, Mongolia

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ABSTRACT

In this paper we propose a generating function method for constructing new two- and three-point iterations with $p(3 \le p \le 8)$ order of convergence. This approach allows us to derive a new family of the optimal order iterative methods that include well known methods as special cases. The necessary and sufficient conditions for pth order convergence of the proposed iterations are given in terms of parameters τ_n and α_n. We also propose some generating functions for τ_n and α_n. We give the extension of a class of optimal fourth-order Jarratt's type iterations with $a \neq \frac{2}{3}$. We develop a unified representation of all optimal eighth-order methods. Several numerical results are given to demonstrate the efficiency and the performance of the presented methods and compare them with some other existing methods.

1. Introduction

Solving a nonlinear equations is important in many mathematical and physical problems. In recent years, a number of higher-order iterative methods have been developed and analyzed on this issue, see [1–11] and references therein. Motivated by the recent results in [11] in this paper, we introduce a generating function method for the construction of new two and three-point iterations with pth order of convergence.

This paper is organized as follows. Section 2 is devoted to the construction of a generated function for the optimal fourth-order method. We then present some choices for the parameters τ_n and α_n. Some iterations are proposed among which some are already well known. In Section 3 we propose a family of optimal eighth-order methods, that include many well-known methods as a particular case. We develop a proper representation of eighth-order methods. We also give necessary and sufficient conditions for three-point iterations to be p order of convergence ($p = 5, 6, 7$). In the last Section 4 we employ the new families of proposed methods to solve some nonlinear equations and compare them with some existing methods.

In our previous paper [11] we have considered two and three-point iterative methods

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \tau_n \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

* Corresponding author.
E-mail addresses: tzhanlav@yahoo.com (T. Zhanlav), chuka@jinr.ru (O. Chuluunbaatar), v_ulzii@yahoo.com (V. Ulziibayar).

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and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad z_n = y_n - \bar{\tau}_n \frac{f(y_n)}{f'(x_n)}, \quad x_{n+1} = z_n - \alpha_n \frac{f(z_n)}{f'(x_n)}. \tag{2}$$

We have proved in [11] the following theorems

Theorem 1. *Assume that f(x) is smooth enough function with a simple root $x^* \in I$ and the initial approximation x_0 is close enough to x^*. Then iterative method (1) has fourth-order of convergence if and only if the parameter τ_n is given by*

$$\tau_n = 1 + \theta_n + 2\theta_n^2 + O(\theta_n^3), \quad \theta_n = \frac{f(y_n)}{f(x_n)}. \tag{3}$$

Theorem 2. *Assume that all assumptions of Theorem 1 are fulfilled. Then the three-point iterative methods (2) has an eighth-order of convergence if and only if the parameters $\bar{\tau}_n$ and α_n are given by*

$$\bar{\tau}_n = 1 + 2\theta_n + \beta\theta_n^2 + \gamma\theta_n^3 + \cdots, \tag{4a}$$

or

$$\tau_n = 1 + \theta_n + 2\theta_n^2 + \beta\theta_n^3 + \gamma\theta_n^4 + \cdots, \quad \left(\bar{\tau}_n = \frac{\tau_n - 1}{\theta_n}\right), \tag{4b}$$

and

$$\alpha_n = 1 + 2\theta_n + (\beta + 1)\theta_n^2 + (2\beta + \gamma - 4)\theta_n^3 + (1 + 4\theta_n)\frac{f(z_n)}{f(y_n)} + O(\theta_n^4). \tag{5}$$

Our approach in [11] is constructive in the sense that it proposes a new way to obtain optimal order iterations (see [11] in details).

2. Construction of optimal fourth-order methods

2.1. Generating function method

We show that Theorems 1 and 2 not only give sufficient conditions for iterations to be $p = 4, 8$ order of convergence, but also allow us to construct new iterations with p order of convergence. Obtaining new optimal methods of order four is still important, because they combine higher-order of convergence and low computational cost. We consider the following choice of the parameter τ_n

$$\tau_n = H(\theta_n), \tag{6}$$

where $H(\theta)$ is a real valued function to be determined properly. Obviously τ_n will satisfy the condition (3) if

$$H(0) = 1, \quad H'(0) = 1, \quad H''(0) = 4. \tag{7}$$

We call the function $H(\theta)$ satisfying conditions (7) generating one for iteration (1).

Construction of the generating function allows us to derive a new optimal order family of iterations. The following theorem is a consequence of Theorem 1.

Theorem 3. *Assume that all assumptions of Theorem 1 are fulfilled. Then the optimal fourth-order two-point iterations (1) are obtained by the generating function (6) satisfying the conditions (7).*

Of course, many different variants of the generating function $H(x)$, satisfying condition (7) are possible. We cite here one simple form of them

$$H(x) = \frac{1 + (1 - m\alpha)x + (2 - m\alpha + \frac{m(m-1)}{2}\alpha^2)x^2 + \omega x^3}{(1 - \alpha x)^m}, \quad \alpha, m, \omega \in \mathbb{R}. \tag{8}$$

The optimal two-point iterations (1) with $\tau_n = H(\theta_n)$ given by (8) include a lot of well-known iterations as a special cases. For example, if $\omega = 0,\ m = 1$ and $\alpha = 2 - b,\ b \in \mathbb{R}$, then (1) leads to King's one [5]. If $\alpha = 0,\ m = 1$ and $\omega = 1$ in (8), then (1) leads to new modification of Potra-Ptak's one [4]. If $\alpha = m = 1$ and $\omega = -1$ in (8), then (1) leads to Maheshwari's one [7]. If $\alpha = 1,\ m = 2$ and $\omega = 0$ in (8), then (1) leads to Chun and Lee's one [2] and so on (see Table 1). More recently, Behl et al. [12] proposed a general class of fourth-order optimal methods that includes the well-known Ostrowski's and King's family as special cases. We note that this general class of optimal fourth-order iterations is also included in our methods with $\tau_n = H(\theta_n)$ given by (8) as a special case. Namely, if $m = 3,\ \alpha$ replaced by $-\alpha$ and $\omega = \alpha^2 + \frac{5}{3}\alpha + \frac{4}{3}$ or $\omega = (1 - \frac{\beta}{6})\alpha^3 + \alpha^2 - 2\alpha$, then the iterations (1) with $\tau_n = H(\theta_n)$ given by (8) reduce to (3.8) and (3.10) in [12], respectively. This shows that our class of optimal fourth-order methods is wider than that of [12]. So, we have obtained an optimal fourth-order convergence family of iterative methods with three degrees of freedom based on the generating function method.

Analogously, one can construct the generating function (6) for the third-order iterations (1). It has form

$$\tau_n = H(\theta_n) \equiv \frac{1 + (1 + \eta)\theta_n}{1 + \eta\theta_n}, \quad \eta \in \mathbb{R}. \tag{9}$$

Table 1
Some choices of generating function $H(x)$.

№	m	α	ω	$H(x)$	β	γ	Name of iterations with function $H(x)$
1	0	∀	0	$1+x+2x^2$	0	0	Bi et al. [1]
2	∀	0	0				
3	0	0	1	$1+x+2x^2+x^3$	1	0	optimal Potra-Ptak (OP4)
4	1	2	0	$\frac{1-x}{1-2x}$	4	8	Wang, Liu [10] Cordero et al. [3]
5	1	$\frac{9}{4}$	0	$\frac{1-\frac{5}{4}x-\frac{1}{4}x^2}{1-\frac{9}{4}x}$	$\frac{9}{2}$	$\frac{81}{8}$	Wang, Liu [10]
6	1	1	0	$\frac{1+x^2}{1-x}$	2	2	Chun, Lee's [2] $\beta=0$
7	2	1	0	$\frac{1-x+x^2}{(1-x)^2}$	3	5	Chun, Lee's [2]
8	1	∀	∀	$1+\frac{x(1+(2-\alpha)x+\omega x^2)}{1-\alpha x}$	$2\alpha+\omega$	$\alpha(2\alpha+\omega)^2$	
9	1	0	∀	$1+x+2x^2+\omega x^3$	ω	0	
10	1	$2-b$	0	$1+\frac{x(1+bx)}{1+(b-2)x}$	$2(2-b)$	$2(2-b)^2$	King's [13], Thukral, Petkovic [14] Sharma and Sharma [9]
11	1	2.5	0	$\frac{1-1.5x-0.5x^2}{1--2.5x}$	5	12.5	Bi et al. [1]
12	1	1	-1	$1+\frac{x(1+x-x^2)}{1-x}$	1	1	Maheshwari [7]

2.2. Two-point iterations with two parameters

Let us consider the following two-point iterative method:

$$y_n = x_n - a\frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \tau_n\frac{f(x_n)}{f'(x_n)}, \tag{10}$$

where τ_n is the iteration parameter to be determined properly.

Note that when $a=1$ the iteration (10) leads to (1). However, in some case the iteration (10) is applied with $a\neq 1$. Therefore, the study of convergence of iteration (10) is also interesting and we prove the following.

Theorem 4. *Assume that all assumptions of Theorem 1 are fulfilled. Then the two-point iteration (10) has a fourth-order of convergence if and only if the parameter τ_n is given by*

$$\tau_n = 1 + \frac{\theta_n-(1-a)}{a^2} + 2\left(\frac{\theta_n-(1-a)}{a^2}\right)^2 + \frac{(1-a)(a+4)}{a^3}\left(1-\frac{f'(y_n)}{f'(x_n)} - \frac{2(\theta_n-(1-a))}{a}\right) + O(f^3(x_n)). \tag{11}$$

Proof. We use Taylor expansion of $f(x_{n+1})$ at point x_n.

$$f(x_{n+1}) = (1-\tau_n)f(x_n) + \frac{f''(x_n)}{2}\tau_n^2\left(\frac{f(x_n)}{f'(x_n)}\right)^2 - \frac{f'''(x_n)}{6}\tau_n^3\left(\frac{f(x_n)}{f'(x_n)}\right)^3 + O(f^4(x_n)).$$

Multiplying by $(f'(x_n))^{-1}$ two-sides of last expression, we get

$$(f'(x_n))^{-1}f(x_{n+1}) = \left(1-\tau_n+\tau_n^2\bar{\theta}_n - \frac{\tau_n^3}{6}(f'(x_n))^{-1}f'''(x_n)\left(\frac{f(x_n)}{f'(x_n)}\right)^2\right)(f'(x_n))^{-1}f(x_n) + O(f^4(x_n)), \tag{12}$$

where

$$\bar{\theta}_n = \frac{(f'(x_n))^{-1}f''(x_n)}{2}\frac{f(x_n)}{f'(x_n)} = O(f(x_n)). \tag{13}$$

We seek for τ_n in the form:

$$\tau_n = 1 + \bar{\theta}_n + c\bar{\theta}_n^2 + d_n + O(f^3(x_n)). \tag{14}$$

Substituting (14) into (12) we obtain

$$(f'(x_n))^{-1}f(x_{n+1}) = \left(-\bar{\theta}_n - c\bar{\theta}_n^2 - d_n + \bar{\theta}_n + 2\bar{\theta}_n^2 - \frac{f'''(x_n)}{6f'(x_n)}\left(\frac{f(x_n)}{f'(x_n)}\right)^2\right)(f'(x_n))^{-1}f(x_n) + O(f^4(x_n)),$$

because of $\tau_n^2 = 1+2\bar{\theta}_n + O(f^2(x_n)),\ \tau_n^3 = 1+O(f(x_n))$.

From this we conclude that

$$f(x_{n+1}) = O(f^4(x_n)), \tag{15}$$

under condition

$$c = 2, \quad d_n = -\frac{1}{6}(f'(x_n))^{-1} f'''(x_n) \left(\frac{f(x_n)}{f'(x_n)} \right)^2. \tag{16}$$

Hence

$$\tau_n = 1 + \bar{\theta}_n + 2\bar{\theta}_n^2 - \frac{1}{6}(f'(x_n))^{-1} f'''(x_n) \left(\frac{f(x_n)}{f'(x_n)} \right)^2 + O(f^3(x_n)). \tag{17}$$

It means that the iteration (10) has a fourth-order convergence if and only if τ_n is given by (17).

Next, we using Taylor expansions of $f(y_n)$ and $f'(y_n)$ at point x_n we get

$$\theta_n = \frac{f(y_n)}{f(x_n)} = (1-a) + a^2 \bar{\theta}_n + a^3 d_n + O(f^3(x_n)), \tag{18a}$$

$$(f'(x_n))^{-1} f'(y_n) = 1 - 2a\bar{\theta}_n - 3a^2 d_n + O(f^3(x_n)). \tag{18b}$$

From this we find $\bar{\theta}_n$

$$\bar{\theta}_n = \frac{1}{a^2}\Big(\theta_n - (1-a) - a^3 d_n \Big) + O(f^3(x_n)). \tag{19}$$

Substituting $\bar{\theta}_n$ given by (19) into (17), we get

$$\tau_n = 1 + \frac{\theta_n - (1-a)}{a^2} + 2\left(\frac{\theta_n - (1-a)}{a^2} \right)^2 + (1-a)\left(1 + \frac{4}{a}\right) d_n + O(f^3(x_n)). \tag{20}$$

Elimination $\bar{\theta}_n$ from (18a) and (18b) gives us

$$d_n = \frac{1}{a^2}\left(1 - (f'(x_n))^{-1} f'(y_n) - \frac{2(\theta_n - (1-a))}{a} \right) + O(f^3(x_n)). \tag{21}$$

If we take into account (21) in (20), we get (11). The converse is evident from (12) and (14). □

Note that Theorem 4 is converted to Theorem 2 when $a = 1$. The iteration (10) with $a \neq 1$ is not optimal although its convergence order is four, because it requires four function evalutions $f(x_n), f(y_n), f'(x_n)$ and $f'(y_n)$ per iteration.

From (13) and (16) we find that

$$d_n = -\omega_2 \bar{\theta}_n^2, \quad \omega_2 = \frac{2}{3} \frac{f'''(x_n) f'(x_n)}{(f''(x_n))^2}. \tag{22}$$

Hence the expression (17) can be written as

$$\tau_n = 1 + \bar{\theta}_n + (2 - \omega_2)\bar{\theta}_n^2 + O(f^3(x_n)). \tag{23}$$

2.3. Optimal Jarratt's type iterations

Another class of optimal fourth-order methods appears when τ_n depends not only on θ_n, but also on $\frac{f'(y_n)}{f'(x_n)}$. The well-known Jarratt optimal fourth-order method [15] and method found in [16] has a form (10) with $a = \frac{2}{3}$ and $\tau_n = \frac{1}{2}(1 + \frac{1}{1+\frac{3}{2}b_n})$ and $\tau_n = 1 - \frac{3}{4}b_n + \frac{9}{8}b_n^2,\ b_n = \frac{f'(y_n)}{f'(x_n)} - 1$, respectively.

It is easy to show that τ_n in these methods satisfies the sufficient fourth-order convergence condition (17).

We consider another family of two-point iterative methods

$$y_n = x_n - a\frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \tau_n \frac{f(x_n)}{f'(x_n)}, \tag{24}$$

where the iteration parameter τ_n is given by

$$\tau_n = \frac{1 - \left(\frac{1}{2a} + m\alpha\right)(\xi_n - 1) + \left(\frac{1}{2a^2} + \frac{m\alpha}{2a} + \frac{m(m-1)}{2}\alpha^2\right)(\xi_n - 1)^2}{(1 + \alpha(\xi_n - 1))^m} + \frac{1}{a^2}\left(1 - \frac{3a}{2}\right)\left(\frac{2(1 - a - \theta_n)}{a} - (\xi_n - 1) \right), \quad \alpha, m \in \mathbb{R}, \tag{25}$$

with

$$\theta_n = \frac{f(y_n)}{f(x_n)} \quad \text{and} \quad \xi_n = \frac{f'(y_n)}{f'(x_n)}. \tag{26}$$

Table 2
Some particular cases of (27).

№	Choices of parameters	τ_n	Methods
1	$m=1,\ \alpha=-\frac{3}{2}$	$\frac{1}{2}\left(1+\frac{1}{1+\frac{3}{2}(\xi_n-1)}\right)$	Jarratt's [15]
2	$m=0$ or $\alpha=0$	$1-\frac{3}{4}(\xi_n-1)+\frac{9}{8}(\xi_n-1)^2$	Petkovic et al. [16], Chun [17]
3	$m=1,\ \alpha=-\frac{3}{4}$	$-1-\frac{3}{4}(1-\xi_n)+\frac{8}{1+3\xi_n}$	Chun [17]
4	$m=2,\ \alpha=-\frac{3}{4}$	$1-\frac{4}{1+3\xi_n}+\left(\frac{4}{1+3\xi_n}\right)^2$	Chun [17]
5	$m=1,\ \alpha=-\frac{1}{2}$	$\frac{2g(\xi_n)}{\xi_n},\ g(1)=1,$	
		$g'(1)=-\frac{1}{4},\ g''(1)=\frac{3}{2}$	Lotfi [18]
6	$m=0$ or $\alpha=0$	$\frac{1}{2}(3-\xi_n)G(\xi_n),\ G(1)=1,$	
		$G'(1)=\frac{3}{4},\ G''(1)=2$	Jaisway [19]
7	$m=0$ or $\alpha=0$	$1+\sum_{j=1}^{4}\alpha_j\xi_n^j,\ \alpha_1=\frac{21}{8}-\alpha_4,$	Kattri, Abbasbandy [20]
		$\alpha_2=-\frac{9}{2}+3\alpha_4,\ \alpha_3=\frac{15}{8}-3\alpha_4$	

Above, we prove that the necessary and sufficient condition for two-point iteration to be fourth-order convergence is (17). Using (18b) and (21) it is easy to show that the parameter τ_n given in (25) satisfies the sufficient fourth-order convergence condition (17).

From (25) we see that τ_n depends in generally not only on ξ_n but also on θ_n. The exception is the case $a=\frac{2}{3}$. In this case the second term in (25) disappears and τ_n depends only on ratio ξ_n and per iteration it requires evaluations of $f(x_n), f'(x_n)$ and $f'(y_n)$. So the two parameter family of iterations (24) is optimal one. When $a=\frac{2}{3}$ the formula (25) leads to

$$\tau_n=\frac{1-\left(\frac{3}{4}+m\alpha\right)(\xi_n-1)+\left(\frac{8}{9}+\frac{3}{4}m\alpha+\frac{m(m-1)}{2}\alpha^2\right)(\xi-1)^2}{(1+\alpha(\xi_n-1))^m},\quad \alpha, m\in\mathbb{R}. \tag{27}$$

The iterations (24) with the generating function given by (27) include many well-known iterations as a particular cases. For example, when $m=0,\ \alpha=-\frac{3}{2}$ the iteration (24) with (27) leads to Jarratt's one [15]. When $m=0$ or $\alpha=0$, then the iteration (24) with (27) leads to the method presented in Chun [17] and Petkovic et al. [16]. When $m=1,\ \alpha=-\frac{3}{4}$ and $m=2,\ \alpha=-\frac{3}{4}$ the iterations (24) with (27) leads to the methods given in Chun [17]. When $m=1$ and $\alpha=-\frac{1}{2}$ the iteration (24) with (27) leads to one found in Lotfi [18] (see Table 2). So our iterations (24), (27) can be considered as a generalization of many optimal fourth-order convergence iterations. The fourth order convergent iterations (24), (25) are not optimal when $a\neq\frac{2}{3}$. Per iteration it requires evaluations of $f(x_n), f(y_n), f'(x_n)$ and $f'(y_n)$.

Remark 1. If, instead of a in (24) and (25), we take

$$a_n=\frac{2}{3}+cf(x_n)\neq 0,\quad c\in\mathbb{R}, \tag{28}$$

then the influence of second term in (25) is neglectible because it has a form $O(f^3(x_n))$.

Thus, the iterations (24) remain in the class of optimal fourth-order convergence methods under choice

$$\tau_n=\frac{1-\left(\frac{1}{2a_n}+m\alpha\right)(\xi_n-1)+\left(\frac{1}{2a_n^2}+\frac{m\alpha}{2a_n}+\frac{m(m-1)}{2}\alpha^2\right)(\xi_n-1)^2}{(1-\alpha(\xi_n-1))^m}, \tag{29}$$

where a_n is determined by (28). Thus, we extend significantly the class of optimal fourth-order methods due to (28), (29).

3. Proper representation of the optimal order three-point iterative methods

Recently, based on optimal fourth-order methods some higher-order, in particular eighth order three-point methods have been proposed for solving nonlinear equations (see Table 1). It is easy to show that $\tau_n=H(\theta_n)$ given by (8) satisfies the condition (4b) with constants

$$\beta=\omega+2m\alpha-\frac{m(m-1)}{2}\alpha^2+\frac{m(m-1)(m-2)}{6}\alpha^3, \tag{30a}$$

$$\gamma=\omega m\alpha+m(m+1)\alpha^2-\frac{(m-1)m(m+1)}{3}\alpha^3+\frac{(m-2)(m-1)m(m+1)}{8}\alpha^4. \tag{30b}$$

In Table 1 we present some function $H(\theta_n)$, satisfying the condition (4b).

The following is a consequence of Theorem 2.

Table 3
The optimal order three-point iterative methods.

№	m	$\alpha_n-(1+4\theta_n)\frac{f(z_n)}{f(y_n)}$	Methods
1	0	$1+2\theta_n+(\beta+1)\theta_n^2+(2\beta+\gamma-4)\theta_n^3$	$\beta=\gamma=1,$ [14] Maheshwari-based $\beta=4,\ \gamma=8,$ Method 1 in [14] see [11] $\beta=3,\ \gamma=4,$ Chun Lee [2]
2	1	$\frac{\beta+1+(6-\gamma)\theta_n+((\beta+1)^2-2(2\beta+\gamma-4))\theta_n^2}{\beta+1-(2\beta+\gamma-4)\theta_n}$	$p=d=\omega=0,$ [1–3,6,8–10,14] $\beta=4,\ \gamma=8,$ Method 2 in [14]
3	2	$\frac{1+2(1-q)\theta_n+\left(\frac{6-\gamma}{2}+(\beta-3)q\right)\theta_n^2}{(1-q\theta_n)^2}$	$p=d=\omega=b=0,$ $\beta=4,\ \gamma=8,\ q=2$ Methods 3 in [14]
4	1	$\frac{2-\theta_n}{6\theta_n^2-5\theta_n+2}$	$p=-3,\ d=0,\ q=\frac{5}{2},$ Maheshwari-based optimal methods [22]
5	1	$\frac{2+(5+2\beta)\theta_n+2(3+\beta)\theta_n^2}{2+(1+2\beta)\theta_n-2\beta\theta_n^2}$	$d=0,\ q=-\frac{1+2p}{2},$ [22]
6	1	$\frac{2\beta-1+2\beta(\beta-2)\theta_n}{2\beta-1+2(\beta^2-4\beta+1)\theta_n+(1+4\beta)\theta_n^2}$	$d=0,\ q=\frac{2(\beta^2-4\beta+1)}{1-2\beta},\ p=\frac{1+4\beta}{1-2\beta},$ King-based optimal methods [23]
7	1	$\frac{1}{1-2\theta_n-\theta_n^2}$	$\beta=4,\ \gamma=8,\ q=2,\ p=1,\ d=\omega=0,$ method 4 in [14]

Theorem 5. *Assume that all assumptions of Theorem 1 are fulfilled. Then the family of three-point iterative methods (2) has an eighth-order of convergence if and only if the parameters* τ_n *and* α_n *are given by (6), (8) and*

$$\alpha_n=\left(H(\theta_n)+\theta_n+(\beta-1)\theta_n^2+(\beta+\gamma-4)\theta_n^3\right)+(1+4\theta_n)\frac{f(z_n)}{f(y_n)}. \tag{31}$$

Proof. It is easy to show that $\tau_n=H(\theta_n)$ satisfies the condition (4b) with parameters β and γ given in Table 1 and α_n given by (31) satisfies (5). Then by Theorem 2 the family of methods (2) has eighth-order of convergence. □

Thus, we propose the families of three-point iterative methods (2) with generating function $\tau_n=H(\theta_n)$. They include many well-known eighth-order methods, as a particular cases (see Table 1). The above mentioned methods differ from one to another only by α_n. Moreover, they have the same asymptotic (5), although these are determined by different formulas. Our approach proposed in [11] is constructive in the sense that it discovers a new way to obtain optimal eighth-order iterations.

From (9) and (31) we see that the parameters τ_n and α_n are expressed through the generating function $H(\theta_n)$. It should be pointed out that, to prove the convergence order of iterations usually computer algebraic systems such as Maple, Mathematica and so on are applied, whereas in our approach we use only easily verifiable sufficient conditions (5) and (5). The expression in brackets in (31) can be approximated by a simple rational function without loss of generality. Then α_n can be represented as

$$\alpha_n=\frac{1-(2-mq)\theta_n+c\theta_n^2+\omega\theta_n^3}{\left(1-\theta_n(d\theta_n^2+p\theta_n+q)\right)^m}+(1+4\theta_n)\frac{f(z_n)}{f(y_n)},\quad q,p,d,m\in\mathbb{R}, \tag{32}$$

where

$$\begin{aligned} c&=\beta+1-m(p+2q+\frac{1}{2}(m-1)q^2),\\ \omega&=(2\beta+\gamma-4)-m\left(d+2p+(\beta+1+(1-m)p)q-(m-1)q^2+\frac{(m-1)(m-2)}{6}q^3\right). \end{aligned} \tag{33}$$

We call the optimal order three-point iterative methods (2) with parameters τ_n and α_n given by (6), (8) and (32) respectively proper representation. It is easy to show that all the well-known optimal order three-point iterative methods can be represented in the proper form uniquely (see [1–3,6,8–11,13,14,21–23] and references therein). It should be mentioned that Wu and Lee in [10] first used proper representation of (2). Thus, by means of (6), (8) and (32) we find a unified representation of all optimal order three-point iterations.

Our families of three-point iterative method (2) with parameters τ_n and α_n given by (6), (8) and (32) include the well-known optimal order methods as a particular case (see Table 3).

Theorem 6. *Assume that all assumptions of Theorem 1 are fulfilled. Then the iterations (2) has a pth order of convergence if and only if the parameters* τ_n *and* α_n *are given by formulas presented in Table 4.*

Table 4
The p-order iterative methods.

p	τ_n	α_n
7	(6), (8)	(34)
6	(6), (8)	(35)
	(9)	(34)
5	(6), (8)	(36)
	(9)	(35)

Table 5
Comparison of various fourth-order convergent iterative methods. The factor q in the brackets denotes 10^q.

$f(x)$	x_0	$\tau_n = 1 + \frac{\theta_n(1+(2-\alpha)\theta_n+\omega\theta_n^2)}{1-\alpha\theta_n}$ $\alpha=0,\ \omega=0$			$\alpha=2-b, b=3,\ \omega=0$			$\alpha=0,\ \omega=1$			$\alpha=1,\ \omega=-1$		
		n	$\lvert x^*-x_n\rvert$	COC	n	$\lvert x^*-x_n\rvert$	COC	n	$\lvert x^*-x_n\rvert$	COC	n	$\lvert x^*-x_n\rvert$	COC
$f_1(x)$	3.6	6	5.86(−642)	4.00	6	5.22(−525)	4.00	6	1.37(−796)	4.00	6	9.58(−686)	4.00
	−1	5	5.84(−713)	4.00	5	5.03(−705)	4.00	5	2.84(−716)	4.00	5	8.77(−721)	4.00
$f_2(x)$	4.86	6	1.11(−513)	4.00	6	7.12(−425)	4.00	6	3.08(−584)	4.00	6	9.41(−607)	4.00
	1.6	5	7.20(−699)	4.00	5	9.05(−661)	4.00	5	2.16(−724)	4.00	5	1.37(−726)	4.00
$f_3(x)$	3.2	6	6.86(−509)	4.00	6	2.87(−449)	4.00	6	8.22(−551)	4.00	6	6.79(−561)	4.00
	2.2	5	1.65(−594)	4.00	5	5.31(−557)	4.00	5	4.37(−620)	4.00	5	6.24(−623)	4.00
$f_4(x)$	−0.8	5	3.99(−638)	4.00	5	2.95(−610)	4.00	5	1.21(−655)	4.00	5	7.72(−657)	4.00
	−0.65	4	1.43(−283)	4.00	4	4.59(−274)	4.00	4	1.25(−289)	4.00	4	7.98(−290)	4.00
$f_5(x)$	4	14	8.81(−881)	4.00	14	4.55(−470)	4.00	13	3.06(−330)	4.00	13	2.05(−402)	4.00
	4.5	18	3.81(−312)	4.00	19	3.01(−488)	4.00	18	2.26(−565)	4.00	18	4.70(−768)	4.00
$f_6(x)$	−4.2	16	5.87(−632)	4.00	16	6.35(−288)	4.00	15	4.20(−262)	4.00	15	1.78(−340)	4.00
	−5.8	26	4.46(−579)	4.00	27	1.49(−548)	4.00	25	6.30(−363)	4.00	25	5.78(−595)	4.00

In Table 4 we used following formulae:

$$\alpha_n = \frac{1+(2-\alpha)\theta_n+(\beta+1-2\alpha)\theta_n^2}{1-\alpha\theta_n} + \frac{f(z_n)}{f(y_n)}, \quad \alpha, \beta \in \mathbb{R}, \tag{34}$$

$$\alpha_n = \frac{1+(2-\xi)\theta_n}{1-\xi\theta_n}, \quad \xi \in \mathbb{R}, \tag{35}$$

$$\alpha_n = \frac{1+\varepsilon_1\theta_n}{1-\varepsilon_2\theta_n}, \quad \varepsilon_1, \varepsilon_2 \in \mathbb{R}. \tag{36}$$

Proof. The parameter τ_n and α_n given by (6), (8), (9) and (34)–(36) will satisfy conditions for τ_n and α_n of Theorem 5 in [11]. Hence, by Theorem 5 in [11], the order of iterations (2) is p. □

4. Numerical experiments

We consider six examples taken from [6]

$$\begin{aligned}
&f_1(x) = x^2 - \exp(x) - 3x + 2, && x^* = 0.25753028543983,\\
&f_2(x) = \sin^2(x) - x^2 + 1, && x^* = 1.40449164885154,\\
&f_3(x) = (x-1)^3 - 1, && x^* = 2,\\
&f_4(x) = \sin(x)\exp(x) + \ln(x^2+1), && x^* = -0.60323197152626,\\
&f_5(x) = \exp(x^2+7x-30) - 1, && x^* = 3,\\
&f_6(x) = x\exp(x^2) - \sin^2(x) + 3\cos(x) + 5, && x^* = -1.20764782713287.
\end{aligned}$$

All numerical calculations were performed using Maple 18 system. Also, to study the convergence of iterations (4a) and (4b), we compute the computational order of convergence (COC) of d_{x_n} using the formula, which was taken from [6]

$$d_{x_n} = \frac{\ln\left(|x_{n+1}-x^*|/|x_n-x^*|\right)}{\ln\left(|x_n-x^*|/|x_{n-1}-x^*|\right)}, \tag{37}$$

where x_{n+1}, x_n, x_{n-1} are three consecutive approximations of iterations. n displayed inTables 5–8 is the number iteration required such that $|x_n - x^*| \le 10^{-250}$. In Tables 5, 7 and 8 the factor q in the brackets denotes 10^q.

Table 6
Some three-point iterative methods.

$H(x)$	$\tau_n = H(\theta_n),\ \alpha_n = (H(\theta_n) + \theta_n + (\beta - 1)\theta_n^2 + (\beta + \gamma - 4)\theta_n^3) + (1 + 4\theta_n)\frac{f(z_n)}{f(y_n)}$.									Methods
	$f(x)$ x_0	$f_3(x)$ 3.2		$f_4(x)$ −0.8		$f_5(x)$ 4.5		$f_6(x)$ −4.2		
	$\alpha, \beta, \gamma, \omega$	n	COC	n	COC	n	COC	n	COC	
	$\alpha = 0$, $\beta = 0$, $\gamma = 0$	4	7.99	3	7.99	13	7.99	12	7.99	Bi et al. [1]
$1 + \frac{x+2(1-\alpha)x^2}{(1-\alpha x)^2}$	$\alpha = 1$, $\beta = 3$, $\gamma = 4$	4	8.00	3	7.99	12	7.99	11	8.00	Chun, Lee [2]
	$\alpha = -1$, $\beta = -5$, $\gamma = 8$	4	7.99	3	7.99	14	8.00	12	7.99	New method
	$\alpha = 2$, $\beta = 4$, $\gamma = 8$, $\omega = 0$	4	8.00	3	7.99	11	7.99	10	8.00	Cordero [3]
	$\alpha = 2.5$, $\beta = 5$, $\gamma = 12.5$, $\omega = 0$	4	8.00	3	7.99	8	8.00	7	8.00	Bi et al. [1]
$1 + \frac{x(1+(2-\alpha)x+\omega x^2)}{1-\alpha x}$	$\alpha = 1$, $\beta = 2$, $\gamma = 2$, $\omega = 0$	4	8.00	3	7.99	13	8.00	11	8.00	Chun, Lee [2], $b = 0$, Thukral, Petkovic [14]
	$\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\omega = -1$	4	8.00	3	7.99	13	8.00	11	8.00	Maheshwari-based methods in [8,14]
	$\alpha = -1$, $\beta = 0$, $\gamma = 0$, $\omega = 2$	4	8.00	3	7.99	13	7.99	12	8.00	New method

Table 7
Some particular cases of (29) and (28) with $c = 0$. The factor q in the brackets denotes 10^q.

Choices of parameters	$f_1(x),\ x_0 = -1$			$f_2(x),\ x_0 = 4.86$			Methods
	n	$\lvert x^* - x_n\rvert$	COC	n	$\lvert x^* - x_n\rvert$	COC	
$m = 1,\ \alpha = -\frac{3}{2}$	5	2.49(−847)	4.00	5	7.73(−363)	4.00	Jarratt's [15]
$m = 0$ or $\alpha = 0$	5	3.76(−688)	4.00	6	1.42(−497)	4.00	Petkovic et al. [16], Chun [17]
$m = 1,\ \alpha = -\frac{3}{4}$	5	1.89(−745)	4.00	6	1.62(−696)	4.00	Chun [17]
$m = 2,\ \alpha = -\frac{3}{4}$	5	7.42(−832)	4.00	6	5.03(−924)	4.00	Chun [17]
$m = 1,\ \alpha = -\frac{1}{2}$	5	6.81(−755)	4.00	6	4.70(−604)	4.00	Lotfi [18]
$m = 0$ or $\alpha = 0$	5	3.76(−688)	4.00	6	1.42(−497)	4.00	Jaisway [19]
$m = 0$ or $\alpha = 0$	5	3.76(−688)	4.00	6	1.42(−497)	4.00	Kattri, Abbasbandy [20]

In Table 5 we present results of various fourth-order methods corresponding to (8) with $m = 1$. In particular, the last three columns correspond to King's, modification of Potra-Ptak's and Maheshwari's methods, respectively.

Table 5 shows the methods: the King's [13] method corresponds to $\omega = 0,\ m = 1$ and $\alpha = 2 - b,\ b \in \mathbb{R}$ in (8), the Potra-Ptak's [3] method corresponds to $\omega = 0,\ m = 1$ and $\omega = 1$ in (8), the Maheshwari's [7,8] method corresponds to $\alpha = m = 1$ and $\omega = -1$ in (8), Chun and Lee's [2] method corresponds to $\alpha = 1,\ m = 2$ and $\omega = 0$ in (8) and so on. Fig. 1 shows the residual of the $f_1(x_n)$ versus first five iteration numbers n for different parameters values.

From Table 6 we see that the convergence order of the proposed families with different parameters and the iteration number n are the same as for all considered methods. But the dynamic behavior of iterations may depend on the concrete choices of parameters [24–27]. Find the optimal choices of parameters is important task from practical point of view and deserve additional study.

The results presented in Table 7 are obtained by the known optimal methods with $a = \frac{2}{3}$, whereas results in Table 8 are obtained by optimal methods with $a \neq \frac{2}{3}$ $(a_n = \frac{2}{3} + cf(x_n))$. From Tables 5 to 8 we see that the COC coincides with theoretical one.

Table 8
Some particular cases of (28) and (29) with $c = -1$. The factor q in the brackets denotes 10^q.

	$f_1(x),\ x_0 = -1$			$f_2(x),\ x_0 = 4.86$			$f_3(x),\ x_0 = 2.2$		
Choices of parameters	n	$\lvert x^* - x_n\rvert$	COC	n	$\lvert x^* - x_n\rvert$	COC	n	$\lvert x^* - x_n\rvert$	COC
$m = 1,\ \alpha = -\frac{3}{2}$	5	3.93(−563)	4.00	7	2.17(−899)	4.00	5	9.56(−758)	4.00
$m = 0$ or $\alpha = 0$	5	1.97(−346)	4.00	6	5.21(−313)	4.00	5	2.70(−725)	4.00
$m = 1,\ \alpha = -\frac{3}{4}$	5	4.18(−473)	4.00	6	3.78(−304)	4.00	5	6.36(−753)	4.00
$m = 2,\ \alpha = -\frac{3}{4}$	5	7.22(−347)	4.00	8	3.78(−380)	4.00	5	2.29(−815)	4.00
$m = 1,\ \alpha = -\frac{1}{2}$	5	1.50(−438)	4.00	7	1.45(−962)	4.00	5	3.02(−739)	4.00
$m = 0$ or $\alpha = 0$	5	1.97(−346)	4.00	6	5.21(−313)	4.00	5	2.70(−725)	4.00
$m = 0$ or $\alpha = 0$	5	1.97(−346)	4.00	6	5.21(−313)	4.00	5	2.70(−725)	4.00

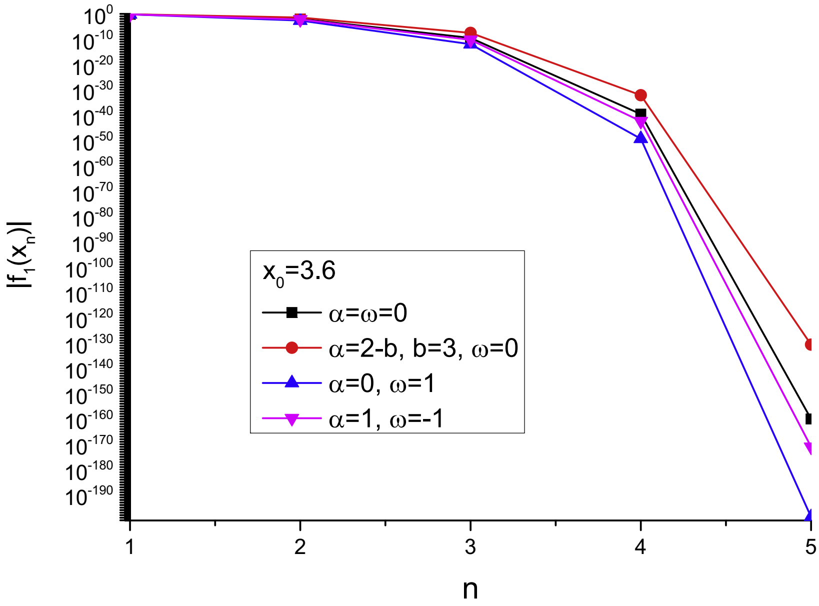

Fig. 1. The residual of the $f_1(x_n)$ versus first five iteration numbers n for different parameters values (see Table 5).

Conclusions

The construction of the generating function for τ_n and α_n allows us to derive new optimal order family of iterations. This family includes many known iterations as a special case. We develop a unified and proper representation of optimal eighth-order three-point methods. The sufficient and necessary conditions for iterations (2) to be p ($p = 5, 6, 7$) order of convergence are also given in term of parameters τ_n and α_n. We also select and propose a new family of two and three-point iterations, for which the parameter τ_n depends not only on θ_n, but also on $\frac{f'(y_n)}{f'(x_n)}$ ratio of first derivatives. For such iterations we also give the necessary and sufficient conditions for the optimal fourth-order of convergence. We extend significantly the class of optimal fourth-order Jarratt's type iterations. It should be mentioned that the dynamic behavior of iterations may depend on the specific choices of parameters. Finding the optimal choices of parameters are an important task from a practical point of view and it deserves an additional analysis.

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