Resonance tunnelling of clusters through repulsive barriers

2014 Phys. Scr. 89 054011

(http://iopscience.iop.org/1402-4896/89/5/054011)

IOP Publishing | Royal Swedish Academy of Sciences Physica Scripta

Phys. Scr. **89** (2014) 054011 (7pp) doi:10.1088/0031-8949/89/5/054011

Resonance tunnelling of clusters through repulsive barriers

A A Gusev[1], S I Vinitsky[1], O Chuluunbaatar[1], A Góźdź[2] and V L Derbov[3]

[1] Joint Institute for Nuclear Research, Dubna, Russia
[2] Institute of Physics, Maria Curie-Skłodowska University, Lublin, Poland
[3] Saratov State University, Saratov, Russia

E-mail: gooseff@jinr.ru

Received 1 November 2013, revised 22 November 2013
Accepted for publication 14 December 2013
Published 29 April 2014

Abstract

Quantum tunnelling of a cluster comprised of several identical particles, coupled via an oscillator-type potential, through short-range repulsive barrier potentials is studied in the s-wave approximation of the symmetrized coordinate representation. A procedure is briefly described that allows the construction of states, symmetric or asymmetric with respect to permutations of A identical particles, from the harmonic oscillator basis functions expressed via the newly introduced symmetrized coordinates. In the coupled-channel approximation of the $\mathcal{R}$-matrix approach, the effect of quantum transparency is analysed; it manifests itself in non-monotonic resonance dependence of the transmission coefficient upon the energy of the particles, their number $A = 3, 4$ and the symmetry types of their states. The total transmission coefficient is shown to demonstrate resonance behaviour with probability density growth in the vicinity of the potential energy local minima, which is a manifestation of the barrier quasi-stationary states, embedded in the continuum.

Keywords: quantum tunnelling, cluster, system of identical particles, permutation symmetry, transmission coefficient, quantum transparency, symmetrized coordinates

(Some figures may appear in colour only in the online journal)

1. Introduction

The mechanism of quantum penetration (tunnelling) of two bound particles through repulsive barriers is a subject of both theoretical and experimental interest in relation to such problems as those of near-surface quantum molecular diffusion, fragmentation in the production of neutron-rich light nuclei, and heavy ion collisions through multidimensional barriers [1–6]. Generalization of the two-particle model over a quantum system of $A > 2$ identical particles is urgently needed for an appropriate description of molecular and heavy ion collisions, as well as for the study of nuclei possessing tetrahedral and octahedral symmetry [7].

Here we consider the penetration of a cluster, consisting of A identical quantum particles, coupled via the short-range oscillator-type potential, through a repulsive potential barrier. We assume that the total spin of the cluster is fixed, so only the coordinate wavefunction is to be considered, which may be symmetric (S) or antisymmetric (A) with respect to the permutation of A identical particles. The initial problem is shown to be reduced to that of the motion of a composite system, with the internal degrees of freedom describing an $(A-1)d$-dimensional oscillator, and the external degrees of freedom describing the cluster centre-of-mass motion in the d-dimensional Euclidean space. For simplicity, we restrict our consideration to the so-called s-wave approximation, in which $d = 1$. The reduction is provided by using appropriately chosen symmetrized coordinates, rather than the conventional Jacobi coordinates. The advantage of the symmetrized coordinates over the Jacobi ones is that they provide invariance of the resulting Hamiltonian with respect to permutations of A identical particles. This, in turn, allows the construction of basis functions that are symmetric or antisymmetric under the permutations not only of $A-1$ relative coordinates, but also of A Cartesian coordinates, i.e., of A real particles that form the cluster. Using this basis to expand the solution is referred to as adopting the symmetrized coordinate representation (SCR).

In the SCR we seek the solution in the form of Galerkin or Kantorovich expansions with unknown coefficients having the form of matrix functions of the centre-of-mass variables. Thus the initial problem is reduced to a boundary-value problem (BVP) for a system of coupled-channel equations in the centre-of-mass variable with conventional asymptotic boundary conditions. The solution procedure involves combined symbolic–numeric algorithms [8, 9]. The results are analysed with particular emphasis on the quantum transparency phenomenon, i.e., the non-monotonic energy dependence of the transmission coefficient, revealing the resonance nature of quantum tunnelling of clusters in S or A states.

2. The setting of the problem

Consider A identical quantum particles having mass m and the set of Cartesian coordinates $x_i \in \mathbf{R}^d$ in the d-dimensional Euclidean space, forming a vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_A) \in \mathbf{R}^{A\times d}$ in the $A \times d$-dimensional configuration space. The particles form a cluster due to the coupling via the pair potential $\tilde{V}^{pair}(\tilde{x}_{ij})$ depending on the relative coordinates $\tilde{x}_{ij} = \tilde{x}_i - \tilde{x}_j$ in a similar way to the potential of a harmonic oscillator $\tilde{V}^{hosc}(\tilde{x}_{ij}) = \frac{m\omega^2}{2}(\tilde{x}_{ij})^2$ with the frequency ω. The particles are considered to penetrate through the repulsive potential barrier $\tilde{V}(\tilde{x}_i)$. Adopting the dimensionless coordinates $x_i = \tilde{x}_i/x_{osc}$, $x_{ij} = \tilde{x}_{ij}/x_{osc} = x_i - x_j$ and energy $E = \tilde{E}/E_{osc}$, $V(x_i) = \tilde{V}(x_i x_{osc})/E_{osc}$, $V^{hosc}(x_{ij}) = \tilde{V}^{hosc}(x_{ij} x_{osc})/E_{osc} = x_{ij}^2/A$, using the oscillator units (osc.u.) $x_{osc} = \sqrt{\hbar/(m\omega\sqrt{A})}$ and $E_{osc} = \hbar\omega\sqrt{A}/2$, one can write the appropriate Schrödinger equation as

$$\left[-\frac{\partial^2}{\partial \mathbf{x}^2} + \sum_{i,j=1;i<j}^{A} \frac{(x_{ij})^2}{A} + U(\mathbf{x}) - E\right]\Psi(\mathbf{x}) = 0,$$
$$U(\mathbf{x}) = \sum_{i,j=1;i<j}^{A} U^{pair}(x_{ij}) + \sum_{i=1}^{A} V(x_i), \qquad (1)$$

where $U^{pair}(x_{ij}) = V^{pair}(x_{ij}) - V^{hosc}(x_{ij})$ is the non-oscillator part of the coupling potential, i.e., if $V^{pair}(x_{ij}) = V^{hosc}(x_{ij})$, then $U^{pair}(x_{ij}) = 0$, $\mathbf{x} = (x_1, \ldots, x_A) \in \mathbf{R}^{A\times d}$.

We seek for the solutions $\Psi(\mathbf{x})$ of equation (1), totally symmetric or antisymmetric under the permutations of A particles that belong to the permutation group S_n. A permutation of particles is nothing but a permutation of the appropriate Cartesian coordinates $x_i \leftrightarrow x_j$, $i, j = 1, \ldots, A$.

The construction of states that retain the symmetry (antisymmetry) under the permutations of A initial Cartesian coordinates (below referred to as S (A) states) is most clearly implemented using the symmetrized relative coordinates rather than the Jacobi ones. One of the possible definitions for the symmetrized coordinates is

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{A-1} \end{pmatrix} = \frac{1}{\sqrt{A}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & a_0 & \cdots & a_0 \\ 1 & a_0 & a_1 & \cdots & a_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_0 & a_0 & \cdots & a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_A \end{pmatrix}, \qquad (2)$$

where $a_0 = 1/(1-\sqrt{A})$, $a_1 = a_0 + \sqrt{A}$. If $A = 2$, then the above symmetrized coordinates are similar to the symmetrized Jacobi coordinates of [10], while for $A = 4$ they correspond to those of [11] (up to a normalizing factor).

With the relations $a_1 - a_0 = \sqrt{A}$, $a_0 - 1 = a_0\sqrt{A}$ taken into account, the relative coordinates $x_{ij} \equiv x_i - x_j$ of a pair of particles i and j are expressed in terms of the internal $A-1$ symmetrized coordinates only:

$$x_{ij} \equiv x_i - x_j = \xi_{i-1} - \xi_{j-1} \equiv \xi_{i-1,j-1}, \qquad (3)$$

$$x_{i1} \equiv x_i - x_1 = \xi_{i-1} + a_0 \sum_{i'=1}^{A-1} \xi_{i'}, \quad i, j = 2, \ldots, A. \qquad (4)$$

In the symmetrized coordinates, equation (1) takes the form

$$\left[-\frac{\partial^2}{\partial \xi_0^2} - \frac{\partial^2}{\partial \boldsymbol{\xi}^2} + \boldsymbol{\xi}^2 + U(\xi_0, \boldsymbol{\xi}) - E\right]\Psi(\xi_0, \boldsymbol{\xi}) = 0,$$
$$U(\xi_0, \boldsymbol{\xi}) = \sum_{i,j=1;i<j}^{A} U^{pair}\left(x_{ij}(\boldsymbol{\xi})\right) + \sum_{i=1}^{A} V(x_i(\xi_0, \boldsymbol{\xi})), \qquad (5)$$

with $\xi_0 \in \mathbf{R}^d$ and $\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_{A-1}\} \in \mathbf{R}^{(A-1)\times d}$, which is invariant under the permutations $\xi_i \leftrightarrow \xi_j$ with $i, j = 1, \ldots, A-1$, i.e., the invariance of equation (1) under the permutations $x_i \leftrightarrow x_j$ with $i, j = 1, \ldots, A$ survives the symmetrizing coordinate transformation (2). This remarkable fact is one of the most prominent features of the proposed approach.

3. The symmetrized coordinate representation

We restrict ourselves to considering $V^{pair}(x_{ij}) = V^{hosc}(x_{ij})$ in the s-wave approximation ($d = 1$). We define the set of SCR cluster functions $\langle \boldsymbol{\xi} | j \rangle^{S(A)} \equiv \Phi_j^{S(A)}(\boldsymbol{\xi})$ and the corresponding energy eigenvalues $\epsilon_j^{S(A)}$ as a solution of the BVP for the equation

$$\left(-\frac{\partial^2}{\partial \boldsymbol{\xi}^2} + \boldsymbol{\xi}^2 - \epsilon_j^{S(A)}\right)\Phi_j^{S(A)}(\boldsymbol{\xi}) = 0. \qquad (6)$$

The solution is sought for in the form of an expansion:

$$\Phi_j^{S(A)}(\boldsymbol{\xi}) = \sum_{\{i_1,\ldots,i_{A-1}\}\in\Delta_j} \alpha_{j[i_1,\ldots,i_{A-1}]}^{S(A)} \Phi_{[i_1,\ldots,i_{A-1}]}^{osc}(\boldsymbol{\xi}). \qquad (7)$$

Here the set $\Delta_j \equiv \{i_1, \ldots, i_{A-1}\}$ is defined by the condition

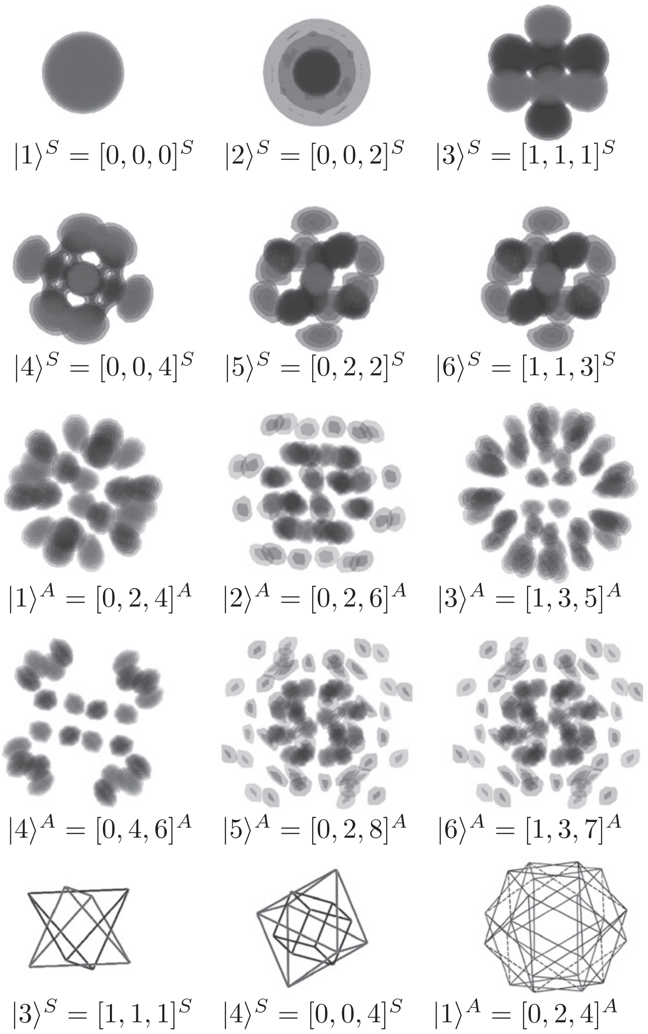

Figure 1. Profiles of the first six 3D oscillator eigenfunctions $\langle \boldsymbol{\xi} \,|j\rangle^{S(A)}$ symmetric and antisymmetric under a permutation of $A = 4$ particles in the internal 3D space (ξ_1, ξ_2, ξ_3). The vertices of the figures illustrate the positions of maxima (black) and minima (grey) for the eigenfunctions $|3\rangle^S$, $|4\rangle^S$, and $|1\rangle^A$.

$\Delta_j = \left\{ i_1, \ldots, i_{A-1} \left| \left(2\sum_{k=1}^{A-1} i_k + A - 1\right) = \epsilon_j^{S(A)} \right. \right\}$, and $\Phi^{osc}_{[i_1,\ldots,i_{A-1}]}(\boldsymbol{\xi}) = \prod_{k=1}^{A-1} \frac{\exp\left(-\xi_k^2/2\right) H_{i_k}(\xi_k)}{\sqrt[4]{\pi}\sqrt{2^{i_k}}\sqrt{i_k!}}$ is the eigenfunction, corresponding to the energy eigenvalue $\epsilon^{osc}_{[i_1,\ldots,i_{A-1}]} \equiv \epsilon^{osc}_f = 2f + A - 1$, $f = \sum_{k=1}^{A-1} i_k$, of the $(A-1)$-dimensional oscillator. The energy levels are degenerate with the degeneracy multiplicity (DM) $p = (A + f - 2)!/f!/(A-2)!$ [12]. The coefficients $\alpha^{S(A)}_{j\,[i_1,\ldots,i_{A-1}]}$ of the orthonormal eigenfunctions $\Phi_j^{S(A)}(\boldsymbol{\xi})$, symmetric (S) (or antisymmetric (A)) under the permutations of A particles and the corresponding eigenvalues $\epsilon_j^{S(A)}$ with the DM $p^{S(A)} \ll p$ are calculated using the SCR algorithm [8] in two steps. First, the eigenfunctions symmetric (or antisymmetric) under the permutations of ξ_i (see equation (3)) are constructed in a standard way. These eigenfunctions are symmetric (antisymmetric) under the permutation of $A - 1$ particles. Second, the eigenfunctions symmetric (antisymmetric) under the permutation of a single pair $x_i \longleftrightarrow x_1$, e.g., $x_2 \longleftrightarrow x_1$ (see equation (4)), are constructed and orthonormalized using the Gram–Schmidt procedure.

In the particular case $A = 3$, $d = 1$, the S (or A) functions can be expressed in polar coordinates $\xi_1 = \rho\cos\varphi$, $\xi_2 = \rho\sin\varphi$ as

$$\Phi^{S(A)}_{k,m}(\rho, \varphi, A = 3) = R_{km}(\rho)\, Y_m^{S(A)}(3m(\varphi + \pi/12)),$$
$$R_{km}(\rho) = \sqrt{2k!/(k + 3m)!}\,(\rho^2)^{3m/2} e^{-\rho^2/2} L_k^{3m}(\rho^2),$$

where $k = 0, 1, \ldots$, and $L_k^{3m}(\rho^2)$ are the generalized Laguerre polynomials [13], $Y_m^S(\varphi) = \cos(\varphi)/\sqrt{(1 + \delta_{m0})\pi}$, $m = 0, 1, \ldots$, for S states, and $Y_m^A(\varphi) = \sin(\varphi)/\sqrt{\pi}$, $m = 1, 2, \ldots$, for A states, that are classified in terms of irreducible representations of the D_{3m} symmetry group. The corresponding energy levels $\epsilon^{S(A)}_{k,m} = 2(2k + 3m + 1)$ have the DM $p^{S(A)} = K + 1$ if the energy $\epsilon^{S(A)}_{k,m} - \epsilon^{S(A)}_{ground} = 12K + K'$, where $K' = 0, 4, 6, 8, 10, 14$, $\epsilon^S_{ground} = 2$ and $\epsilon^A_{ground} = 8$.

For $A = 4$, $d = 1$, the energy levels $\epsilon^{S(A)}_{i_1,i_2,i_3} = 2(i_1 + i_2 + i_3 + 3/2)$ have the DM $p^{S(A)} = 3K^2 + (3 + K')K + K' + \delta_{0K'}$ if the energy $\epsilon^{S(A)}_{i_1,i_2,i_3} - \epsilon^{S(A)}_{ground} = 4(6K + K') + K''$, where $K' = 0, 1, 2, 3, 4, 5$, $K'' = 0, 6$, $\epsilon^S_{ground} = 3$, $\epsilon^A_{ground} = 15$. Here $i_1 = 0, 1, \ldots$, $i_2 = i_1, i_1 + 2, \ldots$, $i_3 = i_2, i_2 + 2, \ldots$ for S states and $i_2 = i_1 + 2, i_1 + 4, \ldots, i_3 = i_2 + 2, i_2 + 4, \ldots$ for A states. The S states with even values of the quantum numbers i_1, i_2, i_3 and the A states with odd ones have the octahedral O_h symmetry, while the A states with even values of i_1, i_2, i_3 and the S states with odd ones have the tetrahedral T_d symmetry. Figure 1 shows example profiles of S and A oscillator eigenfunctions for $A = 4$, $d = 1$. Note that four maxima (black) and four minima (grey) of $|3\rangle^S$ are positioned at the vertices of two tetrahedra forming a *stella octangula*. Eight maxima and six outer minima for $|4\rangle^S$ are positioned at the vertices of a cube and an octahedron. The positions of twelve maxima of $|1\rangle^A$ coincide with the vertices of a polyhedron with 20 triangle faces (only 8 of them being equilateral triangles) and 30 edges, 6 of them having the length 2.25 and the other having the length 2.66 (in oscillator units (osc.u.)).

4. Coupled-channel equations in the SCR

We restrict our consideration to the so-called s-wave approximation ($d = 1$). The asymptotic boundary conditions for the solution $\Psi^{S(A)}(\xi_0, \boldsymbol{\xi}) = \left\{\Psi^{S(A)}_{i_o}(\xi_0, \boldsymbol{\xi})\right\}_{i_o=1}^{N_o}$, describing the incident wave and outgoing waves at $\xi_0^+ \to +\infty$ and

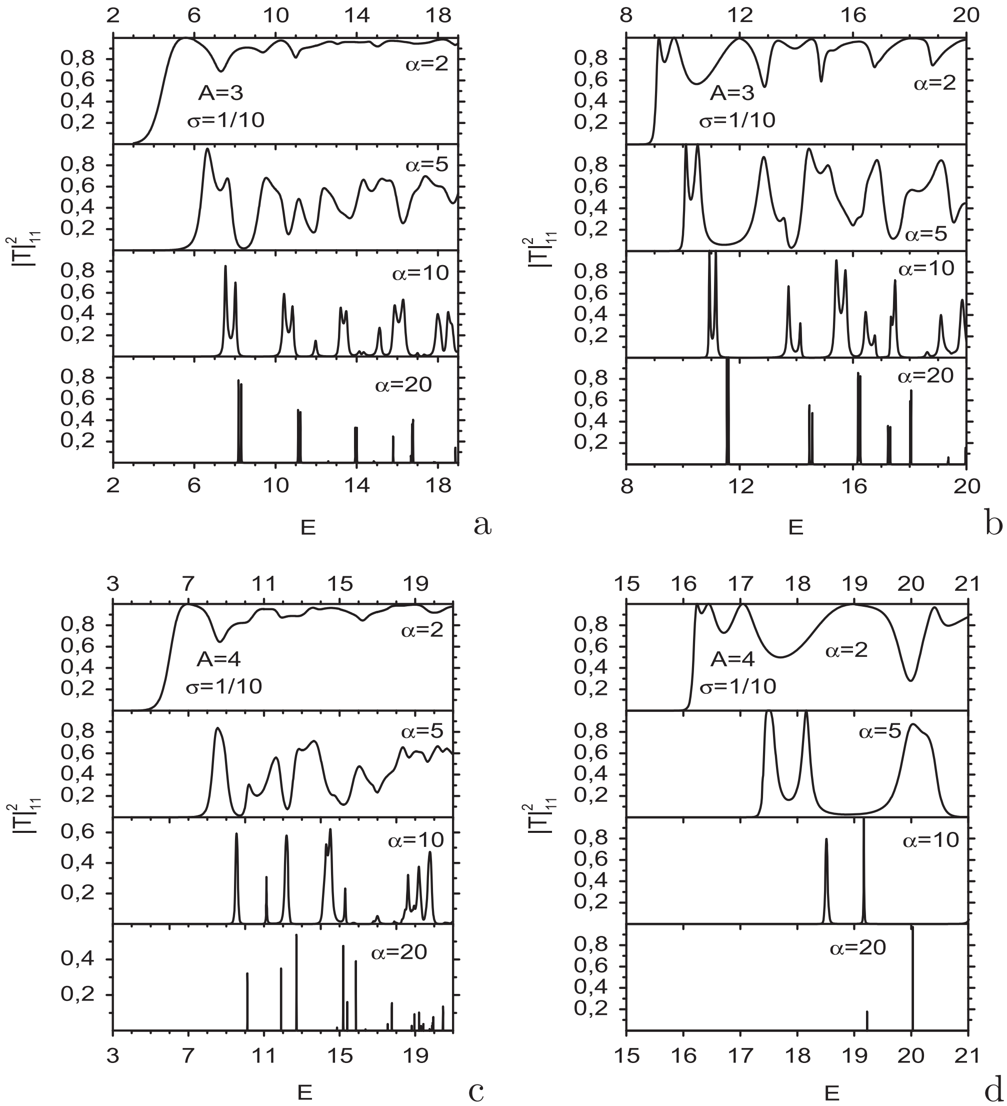

Figure 2. The total probabilities of transmission through the repulsive Gaussian barrier for the system of $A = 3$, 4 particles, coupled by the oscillator potential and initially in the ground symmetric (left) or antisymmetric (right) state, versus the energy E (in osc.u.).

$\xi_0^- \rightarrow -\infty$, can be written in the matrix form $\boldsymbol{\Psi}^{S(A)} = \boldsymbol{\Phi}^{S(A)^T}\boldsymbol{F}$, where

$$\begin{pmatrix} \boldsymbol{F}_\rightarrow(\xi_0^+) & \boldsymbol{F}_\leftarrow(\xi_0^+) \\ \boldsymbol{F}_\rightarrow(\xi_0^-) & \boldsymbol{F}_\leftarrow(\xi_0^-) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{X}^{(-)}(\xi_0^+) \\ \mathbf{X}^{(+)}(\xi_0^-) & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{X}^{(+)}(\xi_0^+) \\ \mathbf{X}^{(-)}(\xi_0^-) & \mathbf{0} \end{pmatrix}\mathbf{S}. \quad (8)$$

Here $\mathbf{X}^{(\mp)}_{i_o}(\xi_0) = \frac{\exp\left(\mp\imath\left(p_{i_o}\xi_0\right)\right)}{\sqrt{p_{i_o}}}$, $v = \leftarrow, \rightarrow$ indicates the initial direction of the particle motion along the ξ_0 axis, and N_o is the number of open channels with the fixed energy E and momentum $p_{i_o}^2 = E - \epsilon_{i_o}^{S(A)} > 0$. The quantities $R_{ji_o}^{\leftarrow} = R_{ji_o}^{\leftarrow}(E)$, $R_{ji_o}^{\rightarrow} = R_{ji_o}^{\rightarrow}(E)$, $T_{ji_o}^{\leftarrow} = T_{ji_o}^{\leftarrow}(E)$, and $T_{ji_o}^{\rightarrow} = T_{ji_o}^{\rightarrow}(E)$ are the unknown amplitudes of the reflected and transmitted waves. $\mathbf{S}$ is the scattering matrix, which is unitary and symmetric [6]:

$$\mathbf{S} = \begin{pmatrix} \mathbf{R}_\rightarrow & \mathbf{T}_\leftarrow \\ \mathbf{T}_\rightarrow & \mathbf{R}_\leftarrow \end{pmatrix}, \quad \mathbf{S}^\dagger\mathbf{S} = \mathbf{S}\mathbf{S}^\dagger = \mathbf{I}. \quad (9)$$

Now we proceed to seek for the solution of the problem (5) in symmetrized coordinates in the form

$$\Psi_{i_o}^{S(A)}(\xi_0, \boldsymbol{\xi}) = \sum_{j=1}^{j_{\max}} \Phi_j^{S(A)}(\boldsymbol{\xi})\chi_{ji_o}^{S(A)}(\xi_0), \quad (10)$$

where $\chi_{ji_o}^{S(A)}(\xi_0)$ are the unknown functions and $\Phi_j^{S(A)}(\boldsymbol{\xi})$ are the SCR cluster functions (7).

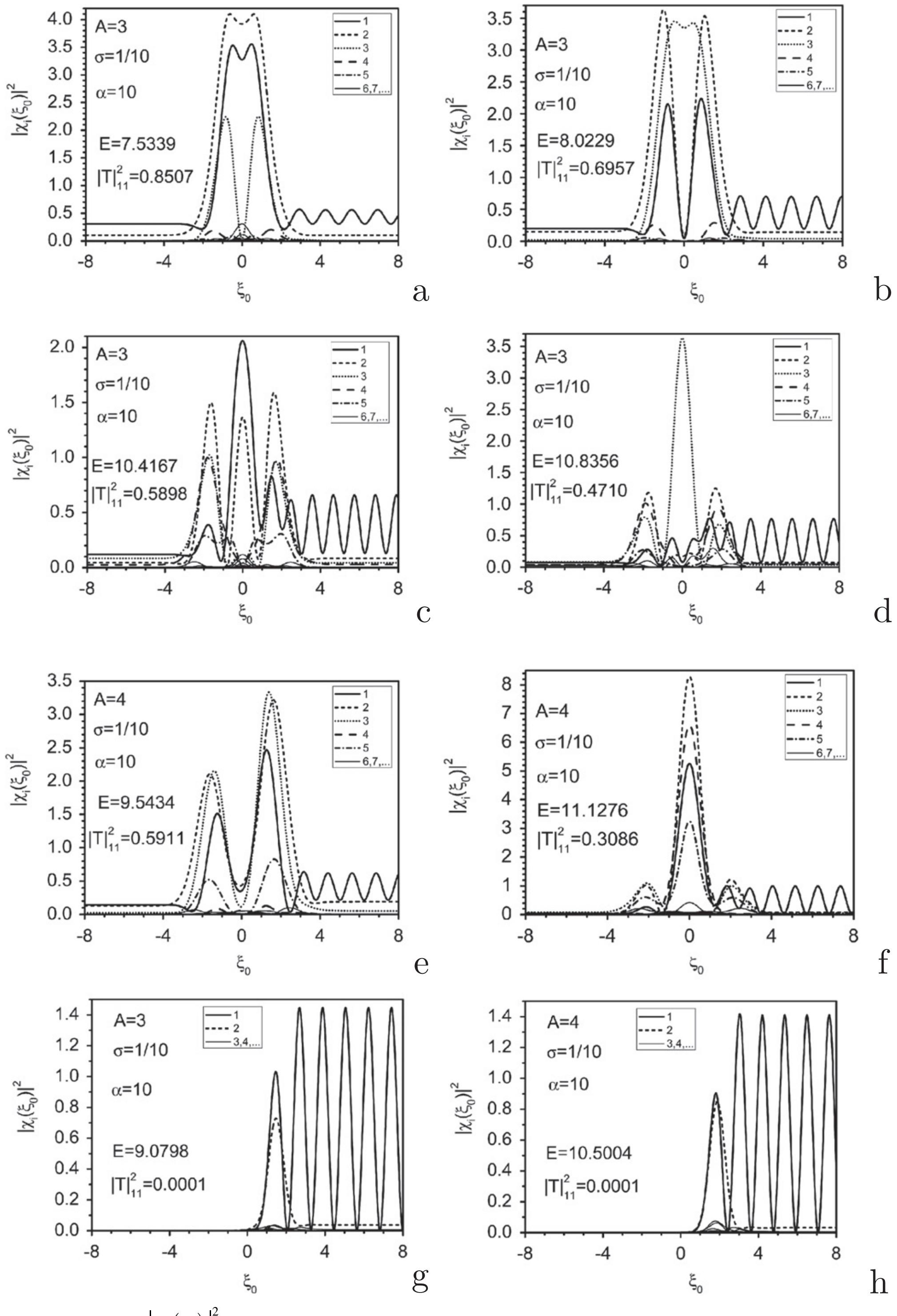

Figure 3. The probability densities $\left|\chi_i(\xi_0)\right|^2$ for the coefficient functions of the decomposition (10), representing the incident wavefunction of the ground S state of the particles at the values of the collision energy E corresponding to individual maxima and minima of the transmission coefficient in figure 2. The parameters of the Gaussian barrier are $\alpha = 10$ and $\sigma = 0.1$.

In the SCR the set of coupled-channel Galerkin-type equations in the centre-of-mass variable has the form

$$\sum_{j=1}^{j_{\max}} \left[\left(-\frac{\mathrm{d}^2}{\mathrm{d}\xi_0^2} - p_i^2 \right) \delta_{ij} + V_{ij}^{S(A)}(\xi_0) \right] \chi_{ji_o}^{S(A)}(\xi_0) = 0, \quad (11)$$

where $V_{ij}^{S(A)}(\xi_0)$ are the effective potentials defined as

$$V_{ij'}^{S(A)}(\xi_0) = \int \mathrm{d}\boldsymbol{\xi}\, \Phi_i^{S(A)}(\boldsymbol{\xi}) \left(\sum_{k=1}^{A} V(x_k) \right) \Phi_{j'}^{S(A)}(\boldsymbol{\xi}). \quad (12)$$

The boundary conditions at $\xi_0 = \xi_t$ and $t = \min, \max$ have

Table 1. Comparison of the resonance energy values E_S and E_A (in osc.u.) for S and A states with the approximate eigenvalues E_i^D, for the first ten quasi-stationary states $i = 1, \ldots, 10$, at $A = 3, 4$ ($\sigma = 1/10$, $\alpha = 20$).

	$A = 3$				$A = 4$			
i	E_S		E_A		E_i^D	E_S	E_i^{D31}	E_i^{D22}
1	8.18	8.31			8.19	10.12	10.03	
2	11.11	11.23			11.09	11.89		11.76
3			11.55	11.61	11.52	12.71	12.60	
4	12.60				12.51	14.86	14.71	
5	13.93	14.00			13.86	15.19	15.04	
6			14.46	14.56	14.42	15.41		15.21
7	14.84	14.88			14.74	15.86		15.64
8	15.79				15.67	16.37	16.18	
9			16.18	16.25	16.11	17.54	17.34	
10	16.67	16.73			16.53	17.76	17.56	

the form

$$\left.\frac{\mathrm{d}\boldsymbol{F}(\xi_0)}{\mathrm{d}\xi_0}\right|_{\xi_0=\xi_t} = \mathcal{R}(\xi_t)\boldsymbol{F}(\xi_t). \tag{13}$$

Here $\mathcal{R}(\xi)$ is an unknown $j_{\max} \times j_{\max}$ $\mathcal{R}$-matrix function and $\boldsymbol{F}(\xi_0) = \{\chi_{i_o}(\xi_0)\}_{i_o=1}^{N_o} = \{\{\chi_{ji_o}(\xi_0)\}_{j=1}^{j_{\max}}\}_{i_o=1}^{N_o}$ is the required $j_{\max} \times N_o\,(j_{\max} \geqslant N_o)$ matrix solution of the BVP (11)–(13) with asymptotes (8)–(9).

5. Results

Consider the repulsive barrier $V(x_i)$ in (12) described by the Gaussian potential $V(x_i) = \frac{\alpha}{\sqrt{2\pi}\sigma}\exp(-\frac{x_i^2}{\sigma^2})$. Figure 2 shows the energy dependence of the total transmission probability $|T|_{ii}^2 = \sum_{j=1}^{N_o}|T_{ji}(E)|^2$. This is the probability of a transition from a chosen state i into any of the N_o states, found from (8)–(9) by solving the BVP (11)–(13) [6]. For this purpose we solve the BVP at $A = 3, 4$: $j_{max} = 21, 39$ ($j_{max} = 16, 15$) for S (A) states with an accuracy of about four significant figures, using the KANTBP program [9] on the finite-element grid $\Omega_\xi\{-\xi_0^{\max}, \xi_0^{\max}\}$, $\xi_0^{\max} = 10.5, 12.8$, with $N_{\text{elem}} = 800, 976$ the fourth-order Lagrange elements between the nodes. Figure 2 illustrates the non-monotonic dependence of the transmission probability upon the energy; the observed resonances are manifestations of the quantum transparency effect. With the barrier height increasing, the peaks become narrower and their positions shift towards higher energies. The multiplet structures of the peaks are similar for symmetric and antisymmetric states.

The effect of quantum transparency, accompanied with the enhancement of the probability density in the vicinity of the potential energy local minima, is due to the existence of barrier quasi-stationary states, embedded in the continuum. Figure 3 shows that in the case of resonance transmission the wavefunctions, depending on the centre-of-mass variable ξ_0, are localized in the vicinity of the potential barrier centre ($\xi_0 = 0$). The correspondence between the probability density distributions shown in figure 3 and the transmission probability features in figure 2 is the following: figures 3(a)–(d) correspond to the first four peaks in figure 2(a), the third panel from the top; figures 3(e), (f) correspond to the first two peaks in figure 2(c), the third panel from the top; figure 3(g) corresponds to the dip between the second and the third peaks in figure 2(a), the third panel from the top; figure 3(h) corresponds to the dip between the first and second peaks in figure 2(c), the third panel from the top.

Table 1 presents the resonance values of the energy E_S (E_A) calculated by solving the BVP (11)–(13) for S (A) states at $A = 3, 4$ $\sigma = 1/10$, $\alpha = 20$, which correspond to the maxima of the transmission coefficients $|T|_{ii}^2$ in figure 2 for $E < 18$, and the corresponding approximate energy eigenvalues E_i^D of the quasi-stationary states of the BVP for equation (1), calculated using the Galerkin sets of 816, 1820 basis functions of the truncated A-dimensional oscillator at $A = 3, 4$, calculated by means of the DC algorithm [8]. The approximation of a narrow barrier with impermeable walls used in the DC algorithm is seen to provide a good approximation E_i^D of the above high-accuracy results E_S and E_A, with the error smaller than 2%.

When $A = 3$ there are six similar wells, three of them on each side of the plane $\xi_0 = 0$. The symmetry with respect to the plane $\xi_0 = 0$ explains the presence of doublets. The presence of states with definite symmetry is associated with the fact that the axis ξ_0 is a third-order symmetry axis.

When $A = 4$ there are 14 wells. Six wells in the centre correspond to the case where two particles are located on one side of the barrier and the other two are on the other side. The corresponding energy eigenvalue is denoted by E_i^{D22}. The remaining eight wells correspond to the case where one particle is located on one side of the barrier and the remaining three are on the other side. The corresponding energy eigenvalue is denoted by E_i^{D31}. For these states, one expects doublets to be observed, like for the case of three particles.

However, the separation between the energy levels is much smaller, because the four-well groups are strongly separated by two barriers, rather than only one barrier as for the case $A = 3$.

6. Conclusion

A cluster model consisting of A identical particles bound by an oscillator-type potential in the external field of a target was formulated in new symmetrized coordinates. Typical examples of clusters with $A = 3$ and $A = 4$ identical particles were analysed in the s-wave approximation, and the correspondence was revealed between the representations of the symmetry groups D_3 for $A = 3$ SCR cluster function shapes, and T_d and O_h for $A = 4$ ones. We demonstrated the quantum transparency effect that manifests itself in a non-monotonic resonance-type dependence of the transmission coefficient upon the energy of the particles, their number $A = 3$, 4, and the symmetry types of their states. We found that this effect accompanies an enhancement of the probability density in the vicinity of local potential minima and is related to the existence of sub-barrier quasi-stationary states, embedded in the continuum.

The proposed approach can be adapted to analyse nuclei having tetrahedral and octahedral symmetry, the quantum diffusion of molecules and microscopic clusters through surfaces, and the fragmentation mechanism involved in the production of neutron-rich light nuclei.

Acknowledgments

The work was supported by grants 13-602-02 JINR, 11-01-00523 and 14-01-00420 RFBR and the Bogoliubov–Infeld program.

References

[1] Hofmann H 1974 *Nucl. Phys.* A **224** 116
[2] Ring R, Rasmussen J and Massman H 1976 *Sov. J. Part. Nucl.* **7** 916
[3] Hagino K, Rowley N and Kruppa A T 1999 *Comput. Phys. Commun.* **123** 143
[4] Pen'kov F M 2000 *JETP* **91** 698
[5] Ahsan N and Volya A 2010 *Phys. Rev.* C **82** 064607
[6] Gusev A *et al* 2011 *Lecture Notes in Computer Science* vol 6885 (Berlin: Springer) p 175
[7] Dobrowolski A *et al* 2011 *Int. J. Mod. Phys.* E **20** 500
[8] Gusev A *et al* 2013 *Lecture Notes in Computer Science* vol 8136 (Berlin: Springer) p 155
Vinitsky S *et al* 2013 *Lecture Notes in Computer Science* vol 8136 (Berlin: Springer) p 427
[9] Chuluunbaatar O *et al* 2008 *Comput. Phys. Commun.* **179** 685 http://wwwinfo.jinr.ru/programs/jinrlib/kantbp/indexe.html
[10] Kamuntavičius G P *et al* 2001 *Nucl. Phys.* A **695** 191
[11] Kramer P and Moshinsky M 1966 *Nucl. Phys.* **82** 241
[12] Baker G A Jr 1956 *Phys. Rev.* **103** 1119
[13] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)