Higher-order Accurate Numerical Solution of Burgers' Equation

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ABSTRACT

Higher-order accurate finite-difference schemes for numerical solution of Burgers' equation which often arises in mathematical modelling used to solve problems in fluid dynamics is presented. The accuracy of the proposed schemes is demonstrated by some test problems. The numerical results are found in good agreement with exact solutions.

Keywords-Burgers' equation, Higher-order accurate numerical solution**.**

1. INTRODUCTION

We consider the one-dimensional quasi-linear parabolic partial differential equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\gamma\frac{\partial^2 u}{\partial x^2},\qquad a<x<b,\qquad t>0, \tag{1}$$

with an initial condition

$$u(x,0)=\varphi(x),\qquad a<x<b, \tag{2}$$

and boundary conditions

$$u(a,t)=f(t)\text{ and }u(b,t)=g(t),\qquad t>0, \tag{3}$$

where $\gamma>0$ is a coefficient of the kinematic viscosity and $\varphi(x)$, $f(t)$ and $g(t)$ are known functions.

The Burgers' equation can be considered as an approach to the Navier-Stokes equations [1, 2]. Since both contain nonlinear terms of the type: unknown functions multiplied by a first derivative and both contain higher-order terms multiplied by a small parameter. On the other hand, the Burgers' equation is one of a few nonlinear equations which can be solved exactly for an arbitrary initial and bounder conditions [3]. However these exact solutions are impractical for the small values of viscosity constant due to a slow convergence of series solutions. Thus many numerical schemes are constructed for a numerical solution of the Burgers' equation for small values of viscosity constant which corresponds to a steep front in the propagation of dynamic wave forms [3-8]. The study of the general properties of the Burgers' equation has motivated considerable attention due to its applications in field as diverse as number theory, gas dynamics, heat conduction, elasticity, etc. [3]. The aim of this paper is to construct higher-order finite-difference schemes for solution of Burgers' equation.

2. STATEMENT OF THE PROBLEM

We consider the Burgers' equation (1) with the initial condition

$$u(x,0)=\sin(\pi x),\quad 0<x<1, \tag{4}$$

and the Dirichlet boundary conditions

$$u(0,t)=u(1,t)=0,\qquad t>0. \tag{5}$$

In is well known that, by the Hopf-Cole transformation

$$u(x,t)=-2\gamma\frac{\theta_x'(x,t)}{\theta(x,t)} \tag{6}$$

the Burgers' equation transforms to the linear heat equation

$$\frac{\partial\theta(x,t)}{\partial t}=\gamma\frac{\partial^2\theta(x,t)}{\partial x^2},\qquad 0<x<1,\qquad t>0 \tag{7}$$

with initial condition

$$\theta(x,0)=\exp\left\{-\frac{1-\cos(\pi x)}{2\pi\gamma}\right\},\qquad 0<x<1 \tag{8}$$

and Neumann boundary conditions

$$\theta_x'(0,t)=\theta_x'(1,t)=0,\qquad t>0. \tag{9}$$

Symbol " ′" denotes the derivative with respect to variable x. Thus, if $\theta(x,t)$ is any solution of the heat equation (7) subject to the conditions (8) and (9), then the function (6) is a solution of the Burgers' equation (1) with the conditions (4) and (5).

The Fourier solution to the above heat problem defined by Eqs. (7)-(9) can obtained easily as [3]

$$\theta(x,t)=a_0+\sum_{n=1}^{\infty}a_n\exp(-n^2\pi^2\gamma t)\cos(n\pi x), \tag{10}$$

with the Fourier coefficients

$$a_0=\int_0^1\theta(x,0)dx,$$

$$a_n=2\int_0^1\theta(x,0)\cos(n\pi x)\,dx,\ \ n=1,2,3\ldots \tag{11}$$

Therefore, the (exact) Fourier solution to the problem given by Eqs. (1), (4) and (5) is obtained as [3]

$$u(x,t)=2\pi\gamma\frac{\sum_{n=1}^{\infty}a_n\exp(-n^2\pi^2\gamma t)\sin(n\pi x)}{a_0+\sum_{n=1}^{\infty}a_n\exp(-n^2\pi^2\gamma t)\cos(n\pi x)}. \quad (12)$$

We assume that the numerical solution of the heat problem defined by Eqs. (7)-(9) is found by any of known methods with higher accuracy. For example, this problem can be solved by well-known Crank-Nicolson scheme [9] and a more accurate explicit scheme, proposed by Zhanlav in [10]

$$\theta_i^{n+1}=\frac{\beta-\alpha}{\beta+\alpha}\theta_i^{n-1}+\frac{\beta}{\beta+\alpha}(\theta_{i-1}^n-2\theta_i^n+\theta_{i+1}^n)+\frac{2\alpha}{\beta+\alpha}\theta_i^n \quad (13)$$

$$\alpha=\frac{2\tau\gamma}{h^2},\qquad i=0,1,\dots,N-1,\qquad Nh=1,\qquad n=1,2,\dots$$

Here and throughout the work, θ_i^n is the approximate solution at the mesh point $(x_i=ih,\ t_i=n\tau)$, where h is a spatial step, τ is the time step. Easy to show that the scheme (13) is stable and its truncation error is of the order $O(\tau^3+h^6)$ provided that

$$\beta=0.2\,,\quad \frac{\tau\gamma}{h^2}=\frac{1}{\sqrt{60}}. \quad (14)$$

When $\beta=1$, the scheme (13) leads to the well-known DuFort-Frankel's one [9].

It should be mentioned that the scheme (13) is a three-level one in time. Hence, in order to find θ_i^n at level two it requires two values θ_i^n at level 0 and 1. i.e., θ_i^0 and θ_i^1. Using the Taylor expansion of $\theta(x,t)$ at point $(x,0)$ and Eq. (7) we obtain

$$\theta(x,t)=\theta(x,0)+\gamma\frac{\partial^2\theta(x,0)}{\partial x^2}\tau+\frac{\gamma^2}{2}\frac{\partial^4\theta(x,0)}{\partial x^4}\tau^2+ +\frac{\gamma^3}{6}\frac{\partial^6\theta(x,0)}{\partial x^6}\tau^3+O(\tau^4) \quad (15)$$

we will find θ_i^1.

3. CONSTRUCTION OF HIGHER-ORDER ACCURATE FINITE-DIFFERENCE SCHEMES

The solution domain $\{(x,t):x\in[0,1],\ t\in(0,\infty)\}$ is discretized into cells described by the node set (x_i,t) in which $x_i=ih, t_n=n\tau, i=0,1,\dots,N,\ Nh=1,\ n=0,1\dots$ We suppose that the solution of Eqs. (7)-(9) is a sufficiently smooth function with respect to x and t. So, from the Taylor expansions of $\theta(x_{i+1},t)$ and $\theta(x_{i-1},t)$ at point (x_i,t) we have

$$\frac{\theta(x_{i+1},t)-\theta(x_{i-1},t)}{2h}=$$

$$=\theta_x'(x_i,t)+\frac{\theta_x'''(x_i,t)}{6}h^2+\frac{\theta_x^{(5)}(x_i,t)}{120}h^4+O(h^6), \quad (16)$$

$$\frac{\theta_x'(x_{i+1},t)-2\theta_x'(x_i,t)+\theta_x'(x_{i-1},t)}{h^2}=$$

$$=\theta_x'''(x_i,t)+\frac{\theta_x^{(5)}(x_i,t)}{12}h^2+O(h^4). \quad (17)$$

Eliminating θ_x''' from (16) and (17) we obtain

$$\frac{\theta(x_{i+1},t)-\theta(x_{i-1},t)}{2h}=\frac{\theta_x'(x_{i+1},t)+4\theta_x'(x_i,t)+\theta_x'(x_{i-1},t)}{6}- -\frac{\theta_x^{(5)}(x_i,t)}{180}h^4+O(h^6). \quad (18)$$

Omitting the small term in the right-hand side of the obtained finite-difference scheme

$$\frac{\theta_{i+1}^n-\theta_{i-1}^n}{2h}=\frac{\theta_{x,i+1}'^n+4\theta_{x,i}'^n+\theta_{x,i-1}'^n}{6},\qquad i=1,2,\dots,N-1 \quad (19)$$

The truncation error of this scheme is $O(h^4)$. Finding θ_x' from (6) and substituting it into (19), we obtain the compact finite-difference scheme for approximate solution $y_i^n\equiv y(x_i,t_n)$ of $u(x_i,t_n)$:

$$\theta_{i-1}^n y_{i-1}^n+4\theta_i^n y_i^n+\theta_{i+1}^n y_{i+1}^n=-\frac{6\gamma}{h}(\theta_{i+1}^n-\theta_{i-1}^n), \quad (20)$$

$$i=1,2,\dots,N-1,\qquad n=1,2,\dots$$

with boundary conditions

$$y_0^n=y_N^n=0. \quad (21)$$

If we denote $\theta_i^n y_i^n$ by v_i^n, then the scheme (18), (19) leads to

$$v_{i-1}^n+4v_i^n+v_{i+1}^n=-\frac{6\gamma}{h}(\theta_{i+1}^n-\theta_{i-1}^n), \quad (22)$$

$$v_0^n=v_N^n=0. \quad (23)$$

The last system has a unique solution set $(v_0^n,v_1^n,\dots,v_N^n)$ since its matrix is diagonally dominant.

It means that the three-diagonal system (20), (21) has a unique solution set $(y_0^n,y_1^n,\dots,y_N^n)$ for each $n=1,2,\dots$, and it can be solved by efficient elimination method [12]. Moreover, it is also possible to obtain a higher accurate finite-difference scheme than (20), (21).

Using the Taylor expansions of $\theta(x_{i+2},t)$ and $\theta(x_{i-2},t)$at the point (x_i,t) we have

$$\frac{\theta(x_{i+2},t)-\theta(x_{i-2},t)}{4h}=$$

$$=\theta_x'(x_i,t)+\frac{\theta_x'''(x_i,t)}{6}4h^2+\frac{\theta_x^{(5)}(x_i,t)}{120}16h^4+O(h^6). \quad (24)$$

We can eliminate the term with $\theta_x^{(5)}(x_i,t)$ from (24) and (16). As a result we have

$$16\frac{\theta(x_{i+1},t)-\theta(x_{i-1},t)}{2h}-\frac{\theta(x_{i+2},t)-\theta(x_{i-2},t)}{4h}=$$

$$=15\theta_x'(x_i,t)+2\theta_x'''(x_i,t)h^2+O(h^6). \quad (25)$$

We also use the well-known five-point approximate formula for $\theta_x'''(x_i,t)$:

$$\theta_i'''=\frac{1}{12h^2}(-\theta_{i-2}'+16\theta_{i-1}'-30\theta_i'+16\theta_{i+1}'-\theta_{i+2}')+O(h^4), \quad (26)$$

which holds for sufficiently smooth function $\theta(x,t)$ with respect to x variable. Substituting (26) into (25) and using the Hope-Cole transformation given by Eq. (6) we obtain

$$v_{i-2}^n - 16v_{i-1}^n - 60v_i^n - 16v_{i+1}^n + v_{i+2}^n = c_i^n, \tag{27}$$

$$c_i^n = -\frac{3\gamma}{h}(32\theta_{i-1}^n - \theta_{i-2}^n - 32\theta_{i+1}^n + \theta_{i+2}^n), \ \ i = 2,3,\dots,N-2.$$

Of course, besides of $v_0^n = v_N^n = 0$ we need additionally two end conditions v_1^n and v_{N-1}^n in order to solve the system (27). We differentiate Eq. (7) $(2k-1)$-times with respect to x and find that

$$\theta_x^{(2k+1)}(x,t) = \frac{1}{\gamma}\frac{\partial}{\partial t}\theta_x^{(2k-1)}(x,t), \ \ k = 1,2,\dots \tag{28}$$

From the Neumann boundary conditions (9) it is obvious that

$$\theta_x^{(2k+1)}(x_0,t) = \theta_x^{(2k+1)}(x_N,t) = 0, \ \ k = 0,1,\dots \tag{29}$$

Then from (16) it follows that

$$\theta(x_1,t) = \theta(x_{-1},t), \ \ x_1 = h, \ \ x_{-1} = -h. \tag{30}$$

Also we differentiate Eq. (6) $(2k)$-times with respect to x and find that

$$\theta_x^{(2k+1)} = -\frac{1}{2\gamma}v_x^{(2k)}(x,t), \ \ \ k = 0,1,\dots \tag{31}$$

If we use (29) in (31) then we obtain

$$v_x^{(2k)}(x_0,t) = v_x^{(2k)}(x_N,t) = 0. \tag{32}$$

From the Taylor expansions

$$v(x_{-1},t) = v(x_0,t) - v_x'(x_0,t)h + \frac{v_x''(x_0,t)}{2}h^2 - \frac{v_x'''(x_0,t)}{6}h^3 + \cdots,$$

$$v(x_1,t) = v(x_0,t) + v_x'(x_0,t)h + \frac{v_x''(x_0,t)}{2}h^2 + \frac{v_x'''(x_0,t)}{6}h^3 + \cdots, \tag{33}$$

$$v(x_{N-1},t) = v(x_N,t) - v_x'(x_N,t)h + \frac{v_x''(x_N,t)}{2}h^2 - \frac{v_x'''(x_N,t)}{6}h^3 + \cdots,$$

$$v(x_{N+1},t) = v(x_N,t) + v_x'(x_N,t)h + \frac{v_x''(x_N,t)}{2}h^2 + \frac{v_x'''(x_N,t)}{6}h^3 + \cdots,$$

and from (32) we conclude that

$$v(x_{-1},t) = -v(x_1,t), \qquad v(x_{N-1},t) = -v(x_{N+1},t). \tag{34}$$

Hence, taking into account (30) and (34), the finite-difference scheme (27) has the forms for $i = 1, N-1$

$$-61v_1^n - 16v_2^n + v_3^n = -\frac{3\gamma}{h}(32\theta_0^n - \theta_1^n - 32\theta_2^n + \theta_3^n), \tag{35}$$

$$v_{N-3}^n - 16v_{N-2}^n - 61v_{N-1}^n = -\frac{3\gamma}{h}(-\theta_{N-3}^n + 32\theta_{N-2}^n + \theta_{N-1}^n - 32\theta_N^n).$$

Thus, we have finite-difference scheme (27), (35) with truncation error $O(h^6)$.

The solution procedure of system (27), (35) is essentially simplified by using Z-folding algorithm [11]. Namely, if we use notation

$$z_i^n = v_{i-1}^n + av_i^n + v_{i+1}^n, \tag{36}$$

then it is easy to show that the Eq. (24) can be re written as

$$z_{i-1}^n + bz_i^n + z_{i+1}^n = c_i^n, \quad i = 2,\dots,N-2 \tag{37}$$

under conditions

$$b = -8 \pm 3\sqrt{14}, \quad a = -8 \mp 3\sqrt{14}. \tag{38}$$

It means, that the solution of pentad-diagonal system (27) leads to three-diagonal systems (37) and (36) consequently, both of which has a diagonally dominance.

Now, we consider z_0 and z_N^n given by (36)

$$z_0^n = v(x_{-1},t_n) + av(x_0,t_n) + v(x_1,t_n). \tag{39}$$

$$z_N^n = v(x_{N-1},t_n) + av(x_N,t_n) + v(x_{N+1},t_n). \tag{40}$$

By (34) and (23) we have

$$z_0^n = z(x_0,t_n) = av(x_0,t_n) = 0,$$

$$z_N^n = z(x_N,t_n) = av(x_N,t_n) = 0 \tag{41}$$

Thus, we obtain the system

$$\begin{aligned} z_{i-1}^n + bz_i^n + z_{i+1}^n = c_i^n, \ \ i = 1,\dots,N-1, \\ z_0^n = z_N^n = 0. \end{aligned} \tag{42}$$

After solving the last system one can solve (36), i.e.

$$\begin{aligned} v_{i-1}^n + av_i^n + v_{i+1}^n = z_i^n, \ \ i = 1,\dots,N-1, \\ v_0^n = v_N^n = 0. \end{aligned} \tag{43}$$

and thereby we obtain

$$u_i^n = \frac{v_i^n}{\theta_i^n}, \ \ i = 0,\dots,N. \tag{44}$$

Thus, we obtain the numerical solution of Burgers' equation (1), (4), (5) with higher accuracy provided that the solution θ_i^n of heat equation (7)-(9) is founded with higher accuracy.

Using (10) and (12) we find the truncation errors $\psi_1(x_i,t)$, $\psi_2(x_i,t)$ of schemes (20) and (27), respectively, as follows

$$\begin{aligned} &\psi_1(x_i,t) = \\ &= v(x_{i-1},t) + 4v(x_i,t) + v(x_{i+1},t) + \frac{6\gamma}{h}\big(\theta(x_{i+1},t) - \theta(x_{i-1},t)\big) = \\ &= 4\pi\gamma\sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\gamma t) n \sin(n\pi x_i) B_{n,1}(\alpha), \end{aligned} \tag{45}$$

$$\begin{aligned} &\psi_2(x_i,t) = v(x_{i-2},t) - 16v(x_{i-1},t) - 60v(x_i,t) - 16v(x_{i+1},t) + \\ &+v(x_{i+2},t) + \frac{3\gamma}{h}\big[32\big(\theta(x_{i-1},t) - \theta(x_{i+1},t)\big) + \theta(x_{i+2},t) - \theta(x_{i-2},t)\big] \\ &= 4\pi\gamma\sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\gamma t) n \sin(n\pi x_i) B_{n,2}(\alpha), \end{aligned} \tag{46}$$

where

$$\begin{aligned} &B_{n,1}(\alpha) = 2 + \cos\alpha - 3\frac{\sin\alpha}{\alpha}, \\ &B_{n,2}(\alpha) = 2\cos^2\alpha - 16\cos\alpha - 31 + 3(16 - \cos\alpha)\frac{\sin\alpha}{\alpha}, \\ &\alpha = \pi nh. \end{aligned} \tag{47}$$

By using the Taylor expansions of functions $\cos\alpha$ and $\sin\alpha$ it is easy to show that

$$B_{n,1}(\alpha) = \frac{\alpha^4}{60}\left(1 - \frac{\alpha^2}{21} + O(\alpha^4)\right) = O(\alpha^4), \tag{48}$$

$$B_{n,1}(\alpha) = -\frac{4}{105}\alpha^6 + \frac{1}{252}\alpha^8 + O(\alpha^{10}) = O(\alpha^6).$$

4. NUMERICAL RESULTS AND CONCLUSIONS

In this section we demonstrate the accuracy of the proposed finite-difference schemes (13), (14), (42), (43) by solving exact solvable problems and compare the numerical results with the existing results. The computations are performed using MatLab.

Example 1. First we consider the Burgers' equation (1), (4), (5). Tables 1, 3 and 5 display convergence of the proposed schemes for the numerical solution $y(x_i,T)$ to the exact solution $u(x_i,T)$ versus the number of nodes N for $\gamma = 1$, $\gamma = 0.1$ and $\gamma = 0.01$ with $T = \frac{1}{10\sqrt{15}}$, $T = \frac{1}{\sqrt{15}}$ and $T = 10\sqrt{15}$. Table 7 presents the numerical results obtained by using the fourth-order weighted scheme [11, 15] and the exact solution for $v = 1$ at $T = \frac{1}{10\sqrt{15}}$. It is clearly observed that both numerical results are reasonably in good agreement with the exact solution. It is seen that for small values of v, one must consider a large of N to obtain proper solution. The maximum absolute error $\|e\|_\infty = \max_{1\le i\le N-1}|u(x_i,T) - y(x_i,T)|$ versus the number of nodes N is displayed in Tables 2, 4, 6 and 8. It gives an approximate rate of convergence of the proposed schemes and the weighted scheme. The errors are consistent with the theoretical expectations of $O(h^6)$ and $O(h^4)$.

Example 2. Consider the Burgers' equation (1) with the Dirichlet boundary conditions (5) and the initial condition

$$u(x,0) = 4x(1-x), \quad 0 < x < 1, \tag{49}$$

The problem (1), (5) and (49) has an exact solution and it is expressed by the formula (12). All needed coefficients a_n are calculated by (11) with

$$\theta(x,0) = \exp\left(\frac{2x^3 - 3x^2}{3\gamma}\right), \quad 0 < x < 1. \tag{50}$$

The higher order numerical and exact solutions of Example 2 for $\gamma = 1$, $\gamma = 0.1$ and $\gamma = 0.001$ with at $T = \frac{1}{\sqrt{15}}$ and $T = 10\sqrt{15}$ are presented in Tables 9-11.

Table 1: Convergence of the proposed schemes for the numerical solution $y(x_i,T)$ to the exact solution $u(x_i,T)$ of Example 1 versus the number of nodes N. Here $\gamma = 1$ and $T = (10\sqrt{15})^{-1}$

	Numerical solution				Exact
x	*N=10*	*N=20*	*N=40*	*N=80*	solution
0.1	0.228649	0.22865030	0.2286503154	0.2286503156451	0.2286503156477
0.2	0.437766	0.43776771	0.4377677345	0.4377677347901	0.4377677347942
0.3	0.608777	0.60877832	0.6087783451	0.6087783454149	0.6087783454190
0.4	0.725196	0.72519674	0.7251967567	0.7251967569572	0.7251967569600
0.5	0.774045	0.77404614	0.7740461512	0.7740461512588	0.7740461512595
0.6	0.747568	0.74756837	0.7475683734	0.7475683733302	0.7475683733289
0.7	0.645016	0.64501619	0.6450161825	0.6450161823893	0.6450161823870
0.8	0.474055	0.47405491	0.4740549069	0.4740549067930	0.4740549067907
0.9	0.251102	0.25110176	0.2511017581	0.2511017580559	0.2511017580546

Table 2: The maximum absolute error $\|e\|_\infty = \max_{1\le i\le N-1}|u(x_i,T) - y(x_i,T)|$ between numerical and exact solutions versus the number of nodes N, and corresponding Runge coefficients.

N	$\|e\|_\infty$	$\|e\|_{\infty h}/\|e\|_{\infty h/2}$
10	1.058410630083717e-006	
20	1.679794564557469e-008	63.008
40	2.635179296994750e-010	63.744
80	4.136968545509490e-012	63.698

Table 3: The same as in Table 1, but $T = (\sqrt{15})^{-1}$ for $\gamma = 0.1$.

	Numerical solution				Exact
x	*N=10*	*N=20*	*N=40*	*N=80*	solution
0.1	0.157856	0.15786370	0.1578638152	0.1578638169663	0.1578638169929
0.2	0.311602	0.31161146	0.3116115963	0.3116115984406	0.3116115984745
0.3	0.456498	0.45650189	0.4565019578	0.4565019588173	0.4565019588320
0.4	0.586299	0.58629360	0.5862935224	0.5862935211388	0.5862935211184
0.5	0.691734	0.69172222	0.6917220370	0.6917220340560	0.691720340080
0.6	0.757624	0.75761273	0.7576125552	0.7576125523766	0.7576125523305
0.7	0.757834	0.75783125	0.7578312008	0.7578312000723	0.7578312000594
0.8	0.650499	0.65050606	0.6505061712	0.6505061729091	0.6505061729349
0.9	0.391645	0.39165314	0.3916532762	0.3916532782669	0.3916532782993

Table 4: The same as in Table 2, but for Table 3.

N	$\|e\|_\infty$	$\|e\|_{\infty h}/\|e\|_{\infty h/2}$
10	1.239220296311849e-005	
20	1.957252674378296e-007	63.314
40	3.066594644884901e-009	63.824
80	4.795597252638117e-011	63.946

Table 5: The same as in Table 1, but $T = 10\sqrt{15}$ for $\gamma = 0.01$.

	Numerical solution			Exact
x	*N=10*	*N=20*	*N=40*	solution
0.1	0.00079050	0.000790454438	0.000790454446504	0.000790454446633
0.2	0.00151237	0.001512270422	0.001512270437967	0.001512270438214
0.3	0.00210061	0.002100472192	0.002100472214085	0.002100472214428
0.4	0.00249817	0.002498001870	0.002498001895787	0.002498001896192
0.5	0.00266108	0.002660896532	0.002660896559381	0.002660896559811
0.6	0.00256436	0.002564189896	0.002564189922944	0.002564189923357
0.7	0.00220777	0.002207614673	0.002207614696226	0.002207614696580
0.8	0.00161958	0.001619472300	0.001619472316504	0.001619472316762
0.9	0.00085680	0.000856738570	0.000856738579092	0.000856738579229

Table 6: The same as in Table 2, but for Table 5.

N	$\|e\|_\infty$	$\|e\|_{\infty h}/\|e\|_{\infty h/2}$
10	1.840226970003904e-007	
20	2.751935918726689e-011	66.87
40	4.300379496946505e-013	63.99

Table 7: The same as Table 1, but for the fourth-order weighted scheme [11, 15] with $\sigma = \frac{1}{2} - \frac{h^2}{12\gamma\tau}$.

	Numerical solution				Exact
x	*N=10*	*N=20*	*N=40*	*N=80*	solution
0.1	0.228658	0.22865086	0.228650350	0.2286503178	0.2286503156
0.2	0.437781	0.43776863	0.437767792	0.4377677383	0.4377677347
0.3	0.608792	0.60877929	0.608778405	0.6087783492	06087783454
0.4	0.725207	0.72519746	0.725196801	0.7251967597	0.7251967569
0.5	0.774050	0.77404642	0.774046168	0.7740461523	0.7740461512
0.6	0.747565	0.74756821	0.747568363	0.7475683726	0.7475683733
0.7	0.645009	0.64501574	0.645016154	0.6450161806	0.6450161823
0.8	0.474047	0.47405441	0.474054875	0.4740549048	0.4740549067
0.9	0.251097	0.25110144	0.251101738	0.2511017568	0.2511017580

Table 8: The same as in Table 2, but for Table 7.

N	$\|e\|_\infty$	$\|e\|_{\infty h}/\|e\|_{\infty h/2}$
10	1.404989751185859e-005	
20	9.509678503549779e-007	14.774
40	6.056235823947986e-008	15.702
80	3.802648973483258e-009	15.926

Table 9: The same as in Table 1, but for Example 2 with $T = (\sqrt{15})^{-1}$ and $\gamma = 1$.

	Numerical solution				Exact
x	*N=10*	*N=20*	*N=40*	*N=80*	solution
0.1	0.02455883	0.02456346	0.02456374	0.02456376	0.02456376
0.2	0.04679722	0.04680605	0.04680659	0.04680662	0.04680663
0.3	0.06459077	0.06460299	0.06460374	0.06460379	0.06460379
0.4	0.07619922	0.07621368	0.07621457	0.07621462	0.07621463
0.5	0.08043556	0.08045088	0.08045183	0.08045188	0.08045189
0.6	0.07680068	0.07681537	0.07681627	0.07681633	0.07681633
0.7	0.06556400	0.06557658	0.06557735	0.06557740	0.06557740
0.8	0.04777049	0.04777969	0.04778025	0.04778029	0.04778029
0.9	0.02516037	0.02516522	0.02516552	0.02516554	0.02516554

Table 10: The same as in Table 9, but for $T = (\sqrt{15})^{-1}$ for $\gamma = 0.1$.

	Numerical solution				Exact
x	*N=10*	*N=20*	*N=40*	*N=80*	solution
0.1	0.16403609	0.16413204	0.16413714	0.16413745	0.16413747
0.2	0.32309362	0.32324755	0.32325614	0.32325666	0.32325669
0.3	0.47179502	0.47195307	0.47196269	0.47196329	0.47196333
0.4	0.60372599	0.60384776	0.60385627	0.60385682	0.60385685
0.5	0.71012965	0.71020720	0.71021370	0.71021413	0.71021416
0.6	0.77704033	0.77709963	0.77710480	0.77710514	0.77710517
0.7	0.77948836	0.77957486	0.77958062	0.77958098	0.77958101
0.8	0.67413875	0.67427966	0.67428743	0.67428790	0.67428793
0.9	0.40982075	0.40995677	0.40996384	0.40996427	0.40996429

Table 11: The same as in Table 9, but for $T = 10\sqrt{15}$ for $\gamma = 0.001$.

	Numerical solution				Exact
x	*N=10*	*N=20*	*N=40*	*N=80*	solution
0.1	0.00269461	0.00257478	0.00256479	0.00256480	0.00256480
0.2	0.00538868	0.00514955	0.00512959	0.00512961	0.00512961
0.3	0.00808135	0.00772426	0.00769438	0.00769441	0.00769441
0.4	0.01077132	0.01029890	0.01025917	0.01025920	0.01025921
0.5	0.01345682	0.01287331	0.01282383	0.01282388	0.01282388
0.6	0.01613521	0.01544610	0.01538696	0.01538702	0.01538702
0.7	0.01879154	0.01799928	0.01793019	0.01793026	0.01793027
0.8	0.02122422	0.02029882	0.02021648	0.02021656	0.02021657
0.9	0.02070041	0.01952431	0.01941856	0.01941867	0.01941868

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