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The Continuous Analogy of Newton's Method for Solving a System of Linear Algebraic Equations

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ABSTRACT

We propose a continuous analogy of Newton's method with inner iteration for solving a system of linear algebraic equations. Implementation of inner iterations is carried out in two ways. The former is to fix the number of inner iterations in advance. The latter is to use the inexact Newton method for solution of the linear system of equations that arises at each stage of outer iterations. We give some new choices of iteration parameter and of forcing term, that ensure the convergence of iterations. The performance and efficiency of the proposed iteration is illustrated by numerical examples that represent a wide range of typical systems.

1. Introduction

We consider a system of linear algebraic equations

$$Ax = f. \tag{1}$$

For a numerical solving of Equation (1) we consider an iterative process:

$$Bv_n = -r_n, x^{(n+1)} = x^{(n)} + \tau_n v_n, n = 0,1,\cdots \tag{2}$$

Of course, the quality of iterative process Equation (2) essentially depends on the choices of matrix B and of iteration parameter $\tau_n \neq 0$.

Let H be a linear space of m-dimensional vectors. We will denote the Euclidean vector norm in H by $\|\cdot\|$, as well as the corresponding norm of matrices.

Theorem 1.1. *Let* $(Av_n, r_n) \neq 0$. *Then a necessary and sufficient condition for the* $\|r_n\|$ *to be decreasing* $(\|r_{n+1}\| < \|r_n\|)$ *is that*

$$\tau_n \in I_n = \begin{cases} \left(0, -\dfrac{2(Av_n, r_n)}{\|Av_n\|^2}\right), & \text{when } (Av_n, r_n) < 0, \\[4mm] \left(-\dfrac{2(Av_n, r_n)}{\|Av_n\|^2}, 0\right), & \text{when } (Av_n, r_n) > 0. \end{cases}$$

Proof. From Equation (2) we obtain

$$r_{n+1} = r_n + \tau_n Av_n.$$

Hence we have

$$\| r_{n+1} \|^2 = \| r_n \|^2 + \tau_n [\tau_n \| Av_n \|^2 + 2(Av_n, r_n)]. \tag{3}$$

The assertion follows from (3).

The interval I_n we call τ-region of convergence of the iteration method (2). Thus we have to choose τ_n from this region. Moreover, it is desirable that the τ_n to be optimal in some sense. Further we will use well-known assertions to study the convergence of (2).

Theorem 1.2. [1]. *Let S be an* $m \times m$ *matrix. Then the successive approximations*

$$x^{(n+1)} = Sx^{(n)} + z, \ n = 0,1,2,\cdots \tag{4}$$

converge for each $z \in R^m$ *and each* $x^{(0)} \in R^m$ *if and only if*

$$\rho(S) < 1, \tag{5}$$

where $\rho(S)$ *is a spectral radius of the matrix S.*

It is easy to show that the iteration process (2) can be rewritten as (4) with iteration matrix

$$S = E - \tau_n B^{-1} A \text{ and } z = \tau_n B^{-1} f. \tag{6}$$

Here E is an $m \times m$ unit matrix.

Theorem 1.3. [2]. *The iteration process* (2) *with parameter* τ_n *given by*

$$\tau_n^* = -\frac{(Av_n, r_n)}{\|Av_n\|^2} \tag{7}$$

converges to x^* *for any* $x^{(0)} \in H$. *The following a*

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posteriori estimate holds true:

$$\|r_n\| = q_{n-1}q_{n-2}\cdots q_0 \|r_0\|, \qquad (8)$$

where

$$q_i = \sqrt{1 - \frac{(Av_i, r_i)^2}{\|Av_i\|^2 \|r_i\|^2}} < 1, \ i = 0,1,\cdots,n-1.$$

We call the nonzero value of τ_n^* defined by (10) the optimal one in the sense that it yields the minimum value of functional $\|r_{n+1}\|$.

2. The Continuous Analogy of Newton's Method

The continuous analogy of Newton's method is also applicable to (1) and leads to [2]

$$Av_n = -r_n, \qquad (9a)$$

$$x^{(n+1)} = x^{(n)} + \tau_n v_n, n = 0,1,\cdots \qquad (9b)$$

It should be mentioned that not only the convergence, but also the convergence rate of iteration (9) depends on the choice of the parameter τ_n. We have the following:

Lemma 2.1. *The sufficient condition for $\|r_n\|$ to be decreasing is* $0 < \tau_n < 2, n = 0,1,\cdots$

Lemma 2.2. *Suppose that $1 < \tau_n < 2$ for all $n \ge 0$. Then the iteration process* (2) *gives two-sided approximations to x^*, i.e.,*

$$x^{(1)} < \cdots < x^{(2m+1)} < x^* < x^{(2m)} < \cdots < x^{(0)} \text{ if } x^* < x^{(0)}$$

or

$$x^{(0)} < \cdots < x^{(2m)} < x^* < x^{(2m+1)} < \cdots < x^{(1)} \text{ if } x^{(0)} < x^*.$$

The proofs are immediately followed from the equalities

$$r_{n+1} = (1-\tau_n)r_n, \ x^{(n+1)} - x^* = (1-\tau_n)\left(x^{(n)} - x^*\right).$$

At each step of iterations one can solve the system (9a) by means of some iterative methods. We call this inner iteration. We consider the following decomposition of A:

$$A = A_1 + A_2,$$

in which the matrix A_1 is simple and invertible. For finding the correction v_n we use inner iteration

$$A_1 v_n^{(e)} = -r_n - A_2 v_n^{(e-1)}, \ v_n^{-1} \equiv 0, \ e = 0,1,\cdots \qquad (10)$$

Theorem 2.3. *Suppose that*

$$\|C\| < 1, \ C = A_2 A_1^{-1}. \qquad (11)$$

Then the inner iteration (10) *converges and holds the following estimate*

$$\left\|v_n - v_n^{(k)}\right\| \le \left\|A_1^{-1}\right\| \frac{\|C\|^{k+1}}{1-\|C\|} \|r_n\|, k = 0,1,\cdots \qquad (12)$$

Proof. The linear system (10) can be rewritten as

$$(E + C) A_1 v_n = -r_n. \qquad (13)$$

By assumption (11) there exists $(E + C)^{-1}$ and the following series representation is valid

$$(E + C)^{-1} = \sum_{j=0}^{\infty} (-1)^j C^j. \qquad (14)$$

Then from (13) it follows that

$$v_n = -A_1^{-1}\left(\sum_{j=0}^{\infty} (-1)^j C^j\right) r_n. \qquad (15)$$

In a similar way, from (10) we have

$$v_n^{(k)} = -A_1^{-1}\left(\sum_{j=0}^{k} (-1)^j C^j\right) r_n. \qquad (16)$$

From (15) and (16) immediately follows (14).

The estimate (12) means that the inner iteration (10) converges under condition (11). In real computations we have to terminate the inner iteration before convergence. We will restrict the number of inner iteration by k. Then the whole iteration process looks like:

$$A_1 v_n^{(l)} = -r_n - A_2 v_n^{(l-1)}, \ v_n^{-1} \equiv 0, l = 0,1,\cdots,k, \qquad (17a)$$

$$x^{(n+1)} = x^{(n)} + \tau_n v_n^{(k)}, \ n = 0,1\cdots \qquad (17b)$$

We now formulate the convergence theorems for these methods.

Theorem 2.4. *Suppose that the condition* (11) *is satisfied and the iteration parameter τ_n is given by*

$$\tau_n = -\frac{\left(Av_n^{(k)}, r_n\right)}{\left\|Av_n^{(k)}\right\|^2}. \qquad (18)$$

Then the iteration process (17) *converges for any $k = 0,1,\cdots$ and for arbitrary chosen starting $x^{(0)}$.*

Proof. We rewrite the iteration process (17) as (2) with

$$B = \left(\sum_{j=0}^{k} (-1)^j C^j\right)^{-1} A_1.$$

By **Theorem 1.3**, such a process converges if we choose τ_n by (7), in which the v_n is replaced by $v_n^{(k)}$.

Obviously the number k may be different for each n. From (12) it is evident, that it suffices to restrict the number of inner iteration only by $k = 0$ or 1, when the residual norm $\|r_n\|$ is small enough.

Theorem 2.5. *Suppose that the condition* (11) *is satisfied. Then the iteration process* (17) *with $\tau_n \in (0,1]$*

converges for any $k = 0, 1, \cdots$ and for arbitrary chosen starting $x^{(0)}$. *The following inequality holds*:

$$\|r_{n+1}\| \le \theta_n \|r_n\|, \quad 0 < \theta_n < 1. \tag{19}$$

Proof. From (17) we get

$$r_{n+1} = r_n + \tau_n A v_n^{(k)}.$$

Using (11) and (16) we rewrite the last expression as

$$r_{n+1} = \left((1 - \tau_n) E + \tau_n (-1)^{k+1} C^{k+1} \right) r_n. \tag{20}$$

If $0 < \tau_n \le 1$, then from (20) we obtain

$$\|r_{n+1}\| \le \theta_n \|r_n\|, \quad \theta_n = 1 - \tau_n \left(1 - \|C\|^{k+1} \right).$$

According to (11), we have $0 < \theta_n < 1$. The convergence of (17) follows from (19).

Corollary 2.6. *Let the condition* (11) *fulfill, and* τ_n *is given by* [3]

$$\tau_n = \min \left\{ \frac{\|r_{n-1}\|}{\|r_n\|} \tau_{n-1}; 1 \right\}, \quad n = 1, 2, \cdots, \quad \tau_0 \approx 0.1.$$

Then the iteration (17) *converges.*

Remark 2.7. *In proofs of Theorems* 2.3-2.5 *the condition* (11) *is essentially used. In particular, it may be fulfilled if the matrix A of the system* (1) *is a strongly diagonally dominant and A_1 is chosen as*

$$A_1 = \operatorname{diag}\{a_{11}, \cdots, a_{mm}\}.$$

Theorem 2.8. *Suppose that the condition* (11) *is satisfied and that the eigenvalues of matrix $D = A_1^{-1} A_2$ are real. Then the spectral radius of iteration matrices of the iterations* (17) *is decreasing with respect to k when* $\tau_n \to 1$.

Proof. The iterations (17) can be rewritten as (4) with iteration matrices

$$S_k = (1 - \tau_n) E + \tau_n (-1)^{k+1} D^{k+1}. \tag{21}$$

We denote by $\lambda(D)$ the eigenvalues of D. If $\tau_n = 1$, then

$$S_k = (-1)^{k+1} D^{k+1}.$$

Therefore, we have

$$\left| \lambda(S_k) \right| = \left| \lambda(D) \right|^{k+1}.$$

By virtue of (11) we obtain $\left| \lambda(C) \right| \le \|C\| < 1$. Since the spectrums of matrices C and D coincide, we also have

$$\left| \lambda(D) \right| < 1.$$

By definition we get

$$\rho(S_k) = \max \left| \lambda(S_k) \right| = \left(\max \left| \lambda(D) \right| \right)^{k+1}$$

$$= \left(\max \left| \lambda(D) \right| \right)^{k} \max \left| \lambda(D) \right|$$

$$< \max \left| \lambda(D) \right|^{k} = \rho(S_{k-1}),$$

which is valid for all $k = 1, 2, \cdots$ Thus we have

$$\rho(S_k) < \rho(S_{k-1}) < \cdots < \rho(S_1) < \rho(S_0) < 1. \tag{22}$$

Obviously, from (21) it is clear that S_k is a continuous function of τ_n. Therefore, (22) is valid for $\tau_n \to 1$.

The iteration (17) with a few small k represents a special interest from a computational view point. Moreover, it is worth to stay at (17) with $k = 0$ in detail. The iteration (17) with $k = 0$ and $A_1 = \operatorname{diag}\{a_{11}, \cdots, a_{mm}\}$ leads to the well-known Jacobi iteration with relaxation parameter $\omega = \tau_n$. It is also known that [1] the Jacobi method with optimal relaxation parameter

$$\omega_{opt} = \frac{2}{2 - \lambda_{\min} - \lambda_{\max}} \tag{23}$$

converges under the assumption that the Jacobi matrix $B = -A_1^{-1} A_2$ has real eigenvalues and spectral radius less than one. Here by $\lambda_{\min}$ and $\lambda_{\max}$ denoted are the smallest and largest eigenvalues of B. Fortunately, we can prove the convergence of Jacobi method with relation parameter under a mild condition than the above mentioned assumption. Namely, we have.

Corollary 2.9. *Suppose that the condition* (11) *is satisfied. Then the Jacobi method with relaxation parameter*:

$$\omega_n = -\frac{\left(A v_n^{(0)}, r_n \right)}{\left\| A v_n^{(0)} \right\|} = \frac{\left((E + C) r_n, r_n \right)}{\left\| (E + C) r_n \right\|^2} \tag{24}$$

converges for any starting.

From (24) it is clear that the relaxation parameter changes depending on n. Therefore the iteration (17) with $k = 0$, and $A_1 = \operatorname{diag}\{a_{11}, \cdots, a_{mm}\}$ and with ω_n given by (24) we call nonstationary Jacobi method with optimal relaxation parameters. The iteration (17) with $k = 0$ and with a lower triangular matrix A_1 leads to Gauss-Seidel with one parameter.

The formula (18) can be rewritten as:

$$\tau_n = \frac{\left(A B^{-1} r_n, r_n \right)}{\left\| A B^{-1} r_n \right\|^2}, \tag{25}$$

where $A B^{-1} = E - (-1)^{k+1} C^{k+1}$. From this it is clear that $\tau_n \to 1$, as $k \to \infty$. This means that whole iteration (17) with the parameter given by (18) converges quadratically in the limit.

Since $\tau_n \to 1$, as $k \to \infty$, then τ-region of convergence for iterations (17) leads to

$$I_n^k = \left(0; -\frac{2\left(Av_n^{(k)}, r_n\right)}{\left\|Av_n^{(k)}\right\|^2}\right) \to (0;2).$$

The number of outer iteration n depends on the number of inner iteration k, *i.e.*, $n = n_k$. In general, n is a decreasing function of k, *i.e.*, $n_{k+1} < n_k$.

On the other hand, the iteration (17) can be considered as a defect correction iteration [1]

$$x^{(n+1)} = x^{(n)} - \tau_n A_{\text{approx}}^{-1} r_n, \left(B^{-1} = A_{\text{approx}}^{-1}\right), \qquad (26)$$

where A_{approx}^{-1} defined by

$$A_{\text{approx}}^{-1} = A_1^{-1}\left(E - C + C^2 + \cdots + (-1)^k C^k\right). \qquad (27)$$

A_{approx}^{-1} is a reasonable approximation to A^{-1} since

$$\left\|A^{-1} - A_{\text{approx}}^{-1}\right\| \le \frac{\left\|A_1^{-1}\right\|\|C\|^{k+1}}{1 - \|C\|} \to 0 \qquad (28)$$

for large k. The choice of parameter τ_n given by (25) allows us to decrease the residual norm from iteration to iteration. By this reason we call (26) as a minimal defect (or residual) correction iteration.

From (26) it follows that

$$r_{n+1} = \left(E - \tau_n A A_{\text{approx}}^{-1}\right) r_n. \qquad (29)$$

From (29) and $A = A_1 + A_2 = (E + C) A_1$, $C = A_2 A_1^{-1}$, it follows that

$$E - \tau_n A A_{\text{approx}}^{-1} = (1 - \tau_n) E + \tau_n (-1)^{k+1} C^{k+1}.$$

Therefore we have

$$\lambda\left(E - \tau_n A A_{\text{approx}}^{-1}\right) = 1 - \tau_n + \tau_n (-1)^{k+1}\left[\lambda(C)\right]^{k+1}.$$

This means that the spectral radius of matrix $E - \tau_n A A_{\text{approx}}^{-1}$ depends on the number of inner iteration k, *i.e.*, $\rho = \rho(k)$. Therefore we have $\rho(k) \to 0$ as $k \to \infty$, because $\tau_n \to 1$ as $k \to \infty$ and from (11)

$$|\lambda(C)| \le \|C\| < 1.$$

Above we considered the cases, where the inner iteration number k is fixed at each outer iteration. It is desirable that the termination criterion for the inner iteration must be chosen carefully to preserve the superlinear convergence rate of the outer iteration. We stay at this problem in more detail. When $\tau \equiv 1$, the iteration (17) is, indeed, inexact Newton (IN) method for (1). Therefore, iteration (17) with parameter τ_n given by formula (18) we call inexact damped Newton method (IDN). According to IN method [4,5], we must choose $\eta_n \in [0,1)$ and continue the inner iteration until satisfy the condition

$$\left\|r_n^{(e)}\right\| \le \eta_{n-1}\|r_n\|, \ r_n^e = Av_n^{(e)} + r_n, \ n = 1,2,\cdots \quad (30)$$

Thus (30) is a stopping criterion for inner iteration (17a). There are several choices of forcing term η_n in inexact Newton method [4-6]. For examples, in [6] two choices of η_n were proposed:

$$\eta_n = \begin{cases} 1 - \eta_{n-1}\alpha_n, & \text{when } \eta_{n-1}\alpha_n < 1, \\ -\dfrac{1 - \eta_{n-1}\alpha_n}{\alpha_n}, & \text{when } \eta_{n-1}\alpha_n \ge 1, \end{cases} \quad (31)$$

with

$$\alpha_n = \frac{\|r_{n-1}\|}{\|r_n\|},$$

and

$$\eta_n = |1 - \tau_n|, \ n = 0,1,\cdots \qquad (32)$$

Since we have formula (18) for τ_n, one can use the second choice (14) with no additional calculations. We can also use formula for $\tau_n = \dfrac{2}{1 + \sqrt{1 + \|r_n\|}}$ [7]. Therefore, according to (32), we get

$$\eta_n = \frac{\sqrt{1 + \|r_n\|} - 1}{\sqrt{1 + \|r_n\|} + 1}. \qquad (33)$$

3. Numerical Results

The quality of the proposed iteration was checked up for numerous examples. We express the matrix A as

$$A = D + A_L + A_U,$$

where $D = \text{diag}\{a_{11}, \cdots, a_{mm}\}$, and A_L, A_U are lower and upper triangular matrices, respectively. All examples are calculated with an accuracy $\|Ax_n - f\| < 10^{-7}$.

The numerical calculations are performed on Acer, CPU 1.8 GHz, 1 GB RAM, and using a software MATLAB R2007a for the Windows XP system.

Example 1. We consider a system of Equation (1) with matrix A and f given by

$$A = \begin{pmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix}_{m \times m}, f = 6 \times \begin{pmatrix} 0.5 \\ 1.0 \\ \vdots \\ 1.0 \\ 0.5 \end{pmatrix}_m,$$

which was solved by the proposed iteration method (17), (18). For comparison, it was solved by Jacobi and successive over relaxation (SOR) iterations with a parameter ω_{opt}.

Example 2.

Table 4. The number *n* of outer iteration with fixed number *k* of inner iteration, and CPU time CT for examples 2 and 3.

A_1		Example 2			Example 3		
		$k = 0$	$k = 1$	$k = 2$	$k = 0$	$k = 1$	$k = 2$
D	*n*	39	17	13	196	89	56
	CT	1.5e–3	9.2e–4	8.6e–4	8.6e–4	6.9e–3	3.1e–3
$A_L + D$	*n*	14	5	13	92	58	41
	CT	6.7e–3	4.9e–4	4.5e–4	3.3e–4	2.7e–3	2.3e–3
SOR	*n*		8			35	
	CT		1.3e–3			5.1e–3	

Table 5. The number *n* of outer iteration, the total number *k* of inner iteration when forcing terms η_n was chosen by formulas (32) and (33), and CPU time CT.

A_1			Example 2	Example 3
D	(33)	CT	1.61e–3	1.07e–2
		n (*k*)	4 (12)	5 (188)
	(32)	CT	1.72e–3	1.08e–2
		n (*k*)	5(11)	5 (135)
$A_L + D$	(33)	CT	1.10e–3	7.53e–3
		n (*k*)	4 (5)	5 (94)
	(32)	CT	1.11e–3	7.42e–3
		n (*k*)	4 (4)	4 (92)

4. Conclusions

Our method with inner iteration is quadratically convergent and therefore it can compete with other iterations such as SOR with an optimal relaxation parameter for a strictly diagonally dominant system. Moreover, our method is also applicable not only for the system with a strictly diagonal dominant matrix, but also for system, the matrix of which is not Hermitian and positive definite.

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Table 6. A comparison of the calculated values *u*(1/2, 1/2) and an exact value (34) versus the dimension *N* of the example 4.

$N + 1$	*u*(1/2, 1/2)
4	0.070312
8	0.072783
16	0.073446
32	0.073615
Exact	0.073671

Table 7. The number *n* of outer iteration with fixed number *k* of inner iteration, and CPU time CT versus the dimension *N* of the example 4.

$N + 1$		$A_1 = D$			$A_1 = A_L + D$			Tridiagonal			SOR
		$k = 0$	$k = 1$	$k = 2$	$k = 0$	$k = 1$	$k = 2$	$k = 0$	$k = 1$	$k = 2$	
4	*n*	64	20	21	32	15	11	39	16	12	11
	CT	2.5e–3	1.3e–3	1.4e–3	1.3e–3	8.8e–4	8.1e–4	1.2e–3	8.3e–4	7.6e–4	1.3e–3
8	*n*	267	60	87	124	60	40	149	40	47	23
	CT	1.4e–2	9.6e–3	1.2e–3	6.7e–3	3.9e–3	4.2e–3	7.9e–3	2.8e–3	3.6e–3	6.1e–3
16	*n*	1010	159	328	476	214	137	546	99	183	46
	CT	5.9e–1	1.4e–1	3.0e–1	2.8e–1	1.7e–1	1.3e–1	2.9e–1	1.0e–1	1.8e–1	1.0e–1

Table 8. The number *n* of outer iteration with unfixed number *k* of inner iteration, the itration parameter τ_n, and CPU time CT versus the dimension *N* of the example 4.

$N + 1$	*n*	$A_1 = D$			$A_1 = A_L + D$			Tridiagonal		
		k	τ_n	CT	*k*	τ_n	CT	*k*	τ_n	CT
4	1	9	1.0304		5	1.0205		5	1.0267	
	2	4	1.0087	3.58e–3	3	0.9845	2.42e–3	2	1.0207	2.22e–3
	3	18	1.0013		10	1.0006		8	1.0037	
8	1	44	1.0235		23	1.0202		23	1.0213	
	2	35	1.0108	3.37e–3	14	1.0118	1.67e–3	18	1.0102	1.77e–2
	3	70	1.0030		34	1.0032		35	1.0034	
16	1	211	1.0122		106	1.0120		106	1.0122	
	2	185	1.0072	6.07e–0	89	1.0075	3.07e–0	93	1.0072	3.10e–0
	3	195	1.0167		99	1.0158		98	1.0165	

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