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Families of Optimal Derivative-Free Two- and Three-Point Iterative Methods for Solving Nonlinear Equations

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Abstract—Necessary and sufficient conditions for derivative-free two- and three-point iterative methods to have the optimal convergence order are obtained. These conditions can be effectively used not only for determining the order of convergence of iterative methods but also for designing new methods. Furthermore, the use of the method of generating functions makes it possible to construct a wide class of optimal derivative-free two- and three-point methods that includes many well-known methods as particular cases. An analytical formula for the optimal choice of the parameter of iterations improving the order of convergence is derived.

Keywords: nonlinear equations, two- and three-point iterations, necessary and sufficient conditions, optimal methods

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1. INTRODUCTION

Presently, there are a lot of iterative methods for solving nonlinear equations and systems of equations (see [1–6]). Among them, there are derivative-free methods, which are helpful if the derivative of the function is difficult or impossible to calculate. The simplest of them are the well-known secant method and Steffensen's method, which have a low order of convergence. Nowadays, we need new optimal methods with the eighth order of convergence because their index of efficiency is $8^{1/4} \approx 1.682$. Such methods have applications in experimental mathematics, number theory, high energy physics, nonlinear simulation, finite element methods used in CAD, 3D graphics, statistics, security, and cryptography (see [7–9]). In the last decade, various derivative-free two- and three-point methods having good convergence properties have been developed (e.g., see [1–33]). The construction of iterative methods with a high order of convergence became possible due to the rapid progress in computing, computer arithmetic, and symbolic computations. In this paper, we propose some families of derivative-free methods based on the method of generating functions proposed in [5] and on the optimal choice of parameters of iterations [6]. A novel direct approach to proving the order of convergence of such methods that does not use symbolic computations is proposed.

The paper is organized as follows. In Section 2, we consider derivative-free two-point iterative methods and obtain necessary and sufficient conditions for these methods to have the fourth order of convergence. The choice of generating functions for the iteration parameter τ is discussed. In particular, optimal finite difference versions of the well-known Kung–Traub, King, and Maheshwari methods are obtained. In Section 3, we consider derivative-free three-point iterative methods and obtain necessary and sufficient conditions for these methods to have the eighth order of convergence. A wide class of optimal three-point iterative methods that includes many known methods as its special cases is proposed. The local convergence of these methods is proved without using symbolic computations. Section 4 presents the results of numerical computations confirming the theoretical results concerning the order of convergence, and these results are compared with the results obtained using other methods.

2. DERIVATIVE-FREE TWO-POINT ITERATIVE METHODS

Consider the derivative-free two-point iterative method

$$y_k = x_k - \frac{f(x_k)}{\phi(x_k)}, \tag{2.1a}$$

$$x_{k+1} = y_k - \overline{\tau}_k \frac{f(y_k)}{\phi(x_k)}, \tag{2.1b}$$

where

$$f'(x) \approx \phi(x) = \frac{f(x+\gamma f(x)) - f(x)}{\gamma f(x)}, \quad \gamma \in R, \tag{2.2}$$

γ is a free nonzero parameter, and $\overline{\tau}_k$ is a parameter to be determined. Here the function $\phi(x) \equiv \phi(x,\gamma)$ depends not only on x but also on the parameter γ; by the definition of derivative, we have

$$f'(x) = \phi(x,\gamma), \quad \gamma \to 0. \tag{2.3}$$

To determine the order of convergence of the iterative method (2.1a), (2.1b), define

$$w_k = \frac{f'(x_k)}{\phi(x_k)} \neq 0. \tag{2.4}$$

Let $f(x) \in C^3(I)$, where I is an interval containing the root x^* of the equation $f(x) = 0$. Then, the Taylor expansions of the functions $f(y_k)$ and $f(x_k + \gamma f(x_k))$ give

$$f(y_k) = (1 - w_k) f(x_k) + \frac{f''(x_k)}{2}\left(\frac{f(x_k)}{f'(x_k)}\right)^2 w_k^2 + O(f^3(x_k)), \tag{2.5}$$

$$\phi(x_k) = f'(x_k)\left(1 + \gamma \frac{f''(x_k)}{2}\frac{f(x_k)}{f'(x_k)}\right) + O(f^2(x_k)). \tag{2.6}$$

Substitute (2.6) into (2.4) to obtain

$$w_k = \frac{1}{1 + \gamma \dfrac{f''(x_k)}{2}\dfrac{f(x_k)}{f'(x_k)}} + O(f^2(x_k)) = 1 - \gamma \frac{f''(x_k)}{2}\frac{f(x_k)}{f'(x_k)} + O(f^2(x_k)), \tag{2.7}$$

or

$$w_k = 1 + O(f(x_k)). \tag{2.8}$$

Taking into account (2.8), we have in (2.5)

$$f(y_k) = O(f^2(x_k)). \tag{2.9}$$

As in [6], we use the notation

$$\theta_k = \frac{f(y_k)}{f(x_k)}. \tag{2.10}$$

Formulas (2.9) and (2.10) imply that $\theta_k = O(f(x_k))$. Using (2.5) in (2.10), we obtain

$$\theta_k = 1 - w_k + \frac{1}{2} w_k \frac{f''(x_k) f(x_k)}{f'(x_k)\phi(x_k)} + O(f^2(x_k)). \tag{2.11}$$

By eliminating $\dfrac{f''(x_k) f(x_k)}{f'(x_k)}$ from (2.7) and (2.11), we obtain

$$w_k^2 - (1 - \gamma\phi(x_k)) w_k - (1 - \theta_k)\gamma\phi(x_k) = O(f^2(x_k)). \tag{2.12}$$

It is seen from (2.12) that w_k depends on θ_k. Due to (2.8), we may seek w_k in the form

$$w_k = 1 - a_k \theta_k + O(f^2(x_k)). \tag{2.13}$$

By substituting (2.13) into (2.12), we obtain

$$\theta_k(\gamma\phi_k - a_k(1+\gamma\phi_k)) = O(f^2(x_k)), \quad \phi_k = \phi(x_k). \tag{2.14}$$

Now (2.14) implies that

$$a_k = \frac{\gamma\phi_k}{1+\gamma\phi_k} + O(f(x_k)). \tag{2.15}$$

By substituting (2.15) into (2.13), we obtain

$$w_k = 1 - \frac{\gamma\phi_k}{1+\gamma\phi_k}\theta_k + O(f^2(x_k)). \tag{2.16}$$

On the other hand, the Taylor expansion of $f(x_{k+1})$ gives

$$f(x_{k+1}) = (1 - \frac{f'(y_k)}{\phi_k}\overline{\tau}_k)f(y_k) + O(f(y_k)^2). \tag{2.17}$$

Due to (2.1a), we have

$$f'(y_k) = f'(x_k)(1 - \frac{f''(x_k)f(x_k)}{f'(x_k)\phi(x_k)}) + O(f^2(x_k)). \tag{2.18}$$

The elimination of the term $\frac{f''(x_k)f(x_k)}{f'(x_k)\phi(x_k)}$ from (12) and (19) yields

$$f'(y_k) = -f'(x_k)\frac{w_k + 2(\theta_k - 1)}{w_k} + O(f^2(x_k)). \tag{2.19}$$

By substituting w_k given by (2.16) into (2.19) and using the expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \quad |x| < 1, \tag{2.20}$$

we obtain

$$f'(y_k) = f'(x_k)\left(1 - \frac{2}{1+\gamma\phi_k}\theta_k\right) + O(f^2(x_k)). \tag{2.21}$$

Using (2.21) in (2.17), we have

$$f(x_{k+1}) = (1 - (1 - \hat{d}_k\theta_k)\overline{\tau}_k)f(y_k) + O(f(y_k)^2), \quad \hat{d}_k = \frac{2+\gamma\phi_k}{1+\gamma\phi_k}. \tag{2.22}$$

Now we can prove the following result.

Theorem 1. *Let $f(x) \in C^3(I)$, and let the initial approximation x_0 be sufficiently close to the simple root $x^* \in I$ of the function $f(x)$. Then, the iterative method* (2.1) *has the fourth order of convergence if and only if the parameter $\overline{\tau}_k$ in* (2.1) *satisfies the condition*

$$\overline{\tau}_k = \frac{1}{1 - \hat{d}_k\theta_k} + O(f^2(x_k)) = 1 + \hat{d}_k\theta_k + O(f^2(x_k)). \tag{2.23}$$

Proof. Suppose that $\overline{\tau}_k$ in (2.1) satisfies condition (2.23). Then

$$1 - (1 - \hat{d}_k\theta_k)\overline{\tau}_k = O(f^2(x_k)),$$

and $f(y_k) = O(f^2(x_k))$ due to (2.8). Therefore, due to (2.22) we have

$$f(x_{k+1}) = O(f(x_k)^4); \tag{2.24}$$

i.e., the order of convergence of (2.1) is four under condition (2.23). Conversely, let method (2.1) have the fourth order of convergence, i.e., let (2.24) hold. Then, (2.24) and (2.22) imply that $f(y_k) = O(f^2(x_k))$ and $1 - (1 - \hat{d}_k\theta_k)\overline{\tau}_k = O(f^2(x_k))$; i.e., $\overline{\tau}_k$ satisfies condition (2.23).

The iterative method (2.4) uses $f(x_k)$, $f(y_k)$, and $\phi(x_k)$ at each iteration step; therefore, it is optimal in the sense of the Kung–Traub conjecture. The second step in (2.1) can be rewritten as

$$x_{k+1} = x_k - \tau_k \frac{f(x_k)}{\phi(x_k)}, \tag{2.25}$$

where

$$\tau_k = 1 + \overline{\tau}_k \theta_k = 1 + \theta_k + \hat{d}_k \theta_k^2 + O(f^3(x_k)). \tag{2.26}$$

If $\phi(x_k, \gamma) = f'(x_k)$ as $\gamma \to 0$, then $w_k = 1$, and formulas (2.23) and (2.26) take the form

$$\overline{\tau}_k = 1 + 2\theta_k + O(f^2(x_k)),$$
$$\tau_k = 1 + \theta_k + 2\theta_k^2 + O(f^3(x_k)),$$

respectively. Thus, the iterative method (2.1) has the form

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \quad x_{k+1} = x_k - \tau_k \frac{f(x_k)}{f'(x_k)}; \tag{2.27}$$

therefore, it is an optimal fourth-order two-point iterative method [6]. As in [5], the generation function method can be applied for constructing new iterative methods (2.1). Certainly, there are various versions of the generation functions $\overline{\tau}_k = H(\theta_k)$ satisfying the conditions

$$H(0) = 1, \quad H'(0) = \hat{d}_k. \tag{2.28}$$

In this paper, we consider the simple form

$$H(x) = \frac{c + (\hat{d}_k c + d)x + \omega x^2}{c + dx + bx^2}, \quad c + d + b \neq 0, \quad c, d, b, \omega \in R. \tag{2.29}$$

We consider some interesting special cases of H.

1. Let $c = 1$, $d = \beta - 2$, and $b = \omega = 0$ in (2.29). Then, we obtain

$$H(x) = \frac{1 + \left(\beta - \frac{\gamma\phi_k}{1 + \gamma\phi_k}\right)x}{1 + (\beta - 2)x}.$$

The iterative method (2.1) with $\overline{\tau}_k = H(\theta_k)$ has the form

$$y_k = x_k - \frac{f(x_k)}{\phi(x_k)}, \quad x_{k+1} = y_k - \frac{1 + \left(\beta - \frac{\gamma\phi_k}{1 + \gamma\phi_k}\right)\theta_k}{1 + (\beta - 2)\theta_k} \frac{f(y_k)}{\phi(x_k)}. \tag{2.30}$$

As $\gamma \to 0$, (2.30) gives the well-known King method. We call (2.30) the finite difference version of the King method.

2. Let $c = b = 1$, $d = -2$, and $\omega = 0$ in (2.29). Then, we obtain

$$H(x) = \frac{1 - \frac{\gamma\phi_k}{1 + \gamma\phi_k} x}{(1 - x)^2}.$$

The iterative method (2.1) with $\overline{\tau}_k = H(\theta_k)$ has the form

$$y_k = x_k - \frac{f(x_k)}{\phi(x_k)}, \quad x_{k+1} = y_k - \frac{1 - \frac{\gamma\phi_k}{1 + \gamma\phi_k}\theta_k}{(1 - \theta_k)^2} \frac{f(y_k)}{\phi(x_k)}. \tag{2.31}$$

As $\gamma \to 0$, (2.31) gives the well-known fourth-order Kung–Traub method. For this reason, we call (2.31) the finite difference version of the Kung–Traub method.

3. Let $c = 1$, $\omega = d = -1$, and $b = 0$ in (2.29). Then, we obtain

$$H(x) = \frac{1 + \frac{1}{1+\gamma\phi_k}x - x^2}{1 - x}.$$

The iterative method (2.1) with $\overline{\tau}_k = H(\theta_k)$ has the form

$$y_k = x_k - \frac{f(x_k)}{\phi(x_k)}, \quad x_{k+1} = y_k - \frac{1 + \frac{1}{1+\gamma\phi_k}\theta_k - \theta_k^2}{1-\theta_k}\frac{f(y_k)}{\phi(x_k)}. \tag{2.32}$$

As $\gamma \to 0$, (2.32) gives the Maheshwari method. For this reason, we call (2.32) the finite difference version of the Maheshwari method.

Note that an attempt to construct derivative-free versions of the Kung and Traub methods was made in [30]. However, the method obtained in [30] differs from our extensions (2.30) and (2.31).

Thus, using the generating function method, we obtain a wide class of optimal derivative-free two-point methods (2.1) with $\overline{\tau}_k = H(\theta_k)$ specified by (2.29). This class has five parameters $(\gamma, c, d, b, \omega)$. The coefficients in (2.29) can depend on the iteration index k. Note that many derivative-free two-point methods were constructed in [1, 2, 7, 14–18]. The class of iterative methods (2.1) proposed in this paper, which is specified by formula (2.29) with the parameter $\overline{\tau}_k$, includes some well-known iterative methods as special cases. Some of them are listed in Table 1. Only $\overline{\tau}_k$ in Ren's method [16, 34] does not belong to the class $H(\theta_k)$ given by (2.29). Thus, the proposed family (2.1) with the parameter specified by (2.29) is a considerable generalization of the methods described in [2, 7, 9, 11–18, 20–23, 26, 27].

The two-point iterative method (2.1) includes one free nonzero parameter γ. It is well known that the convergence can be accelerated by a proper variation of the free parameter $\gamma = \gamma_k$ at each iteration step. This approach is helpful for constructing high order iterative methods with memory (see [9, 22, 23, 25]). We now try to find the optimal free parameter from the accuracy viewpoint. Consider the Taylor expansion of the function $f(\eta_k) = f(x_k + \gamma f(x_k))$ in the neighborhood of x_k

$$f(\eta_k) = (1 + \gamma f'(x_k))f(x_k) + \frac{f''(x_k)}{2}\gamma^2 f^2(x_k) + O(f^3(x_k)). \tag{2.33}$$

Hence, we see that at each step γ can be chosen as

$$\gamma_k = -\frac{1}{f'(x_k)}. \tag{2.34}$$

Then, (2.33) takes the form

$$f(\eta_k) = \frac{f''(x_k)}{2}\frac{f^2(x_k)}{f'(x_k)^2} + O(f^3(x_k)), \tag{2.35}$$

where

$$\eta_k = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{2.36}$$

Therefore, due to (2.35) η_k specified by formula (2.36) can be considered as a new approximation that is better than x_k (2.35). Taking into account (2.34) and (2.35), formulas (2.5) and (2.7) can be written as

$$f(y_k) = (1 - w_k)f(x_k) + f(\eta_k)w_k^2 + O(f^3(x_k)), \tag{2.37}$$

$$w_k = 1 + \frac{f(\eta_k)}{f(x_k)} + O(f^2(x_k)), \tag{2.38}$$

respectively. Substitute (2.38) into (2.37) and use (2.35) to obtain

$$f(y_k) = O(f^3(x_k)). \tag{2.39}$$

Table 1. Iterative methods

	Methods	$\overline{\tau}_k$	Special cases of H determined in (30)
1.	Methods described in [18, 20] and $h(t,s) = (1+t)(1+s)$ [2], $P1$ in [21]	$1 + \hat{d}_k\theta_k$	$c \neq 0,\ d = b = \omega = 0$
2.	Methods described in [10, 15, 12] and $h(t,s) = \frac{1}{1-t-s}$ [2], CTM in [26]	$\frac{1}{1-\hat{d}_k\theta_k}$	$c = 1,\ d = -\hat{d}_k,\ b = \omega = 0$
3.	$h(t,s) = \frac{1+t}{1-s}$ [2, 34]	$\frac{1+\gamma\phi_k + (\hat{d}_k(1+\gamma\phi_k) - 1)\theta_k}{1+\gamma\phi_k - \theta_k}$	$c = 1+\gamma\varphi_k,\ d = -1,\ b = \omega = 0$
4.	Methods described in [7]	$\frac{1+\phi_k + (\hat{d}_k(1+\phi_k) - (2-\phi_k))\theta_k}{1+\phi_k - (2+\phi_k)\theta_k}$	$c = 1+\varphi_k,\ d = -(2+\varphi_k)$, $b = -1,\ \omega = 0$
5.	Chebyshev–Halley family [9]	$\frac{1+\left(\hat{d}_k - \left(2\alpha + \frac{1}{1+\gamma\phi_k}\right)\right)\theta_k + \omega\theta_k^2}{1-\left(2\alpha+\frac{1}{1+\gamma\phi_k}\right)\theta_k + \frac{2\alpha}{1+\gamma\phi_k}\theta_k^2}$	$c = 1,\ d = -\left(2\alpha + \frac{1}{1+\gamma\phi_k}\right)$, $b = \frac{2\alpha}{1+\gamma\phi_k},\ \omega = H(\theta_k)$
6.	Kung–Traub'smethod and method in [11]	$\frac{1}{1-d_k\theta_k + \frac{1}{1+\gamma\phi_k}\theta_k^2}$	$c = 1,\ d = -\hat{d}_k,\ b = \frac{1}{1+\gamma\phi_k}$, $\omega = 0$
7.	Potra–Ptak's method [13, 22]	$1 + d_k\theta_k + \frac{d_k a}{2}\theta_k^2$	$c = 1,\ d = b = 0,\ \omega = \frac{\hat{d}_k a}{2}$
8.	King-type method [23]	$\frac{1+(\gamma-1)\theta_k - \gamma\theta_k^2}{1+\left(\gamma-2-\frac{1}{1-\beta\phi_k}\right)\theta_k + \frac{\gamma-2}{1-\beta\phi_k}\theta_k^2}$	$c = 1,\ d = \gamma - 2 - \frac{1}{1-\beta\phi_k}$, $b = \frac{2-\gamma}{1-\beta\phi_k},\ a = -\gamma$
9.	Methods described in [17]	$\frac{1-\frac{\gamma\phi_k}{1+\gamma\phi_k}\theta_k}{1-2\theta_k+\theta_k^2}$	$c = b = 1,\ d = -2,\ \omega = 0$
10.	Methods described in [16]	$\frac{1}{1-\hat{d}_k\theta_k + a\frac{1+\phi_k}{\phi_k}f^2(x_k)}$	Does not belong to (30)
11.	Methods $P2$ described in [21]	$\frac{1+\theta_k}{1-\frac{\theta_k}{1+\gamma\phi_k}}$	$c = 1,\ d = -\frac{1}{1+\gamma\phi_k},\ b = \omega = 0$
12.	Methods described in [27]	$\frac{1+\phi_k + 2\theta_k}{1+\phi_k - \phi_k\theta_k}$	$c = 1+\phi_k,\ d = -\phi_k,\ b = \omega = 0$
13.	Methods described in [16]	$1 + \hat{d}_k\theta_k + \left(\alpha_1 + \frac{\alpha_2}{(1+\gamma\phi_k)^2}\right)\theta_k^2$	$c = 1,\ d = b = 0,\ \omega = \alpha_1 + \frac{\alpha_2}{(1+\gamma\phi_k)^2}$

Let the parameter $\overline{\tau}_k$ be chosen by the formula

$$\overline{\tau}_k = \frac{1}{1-\hat{d}_k\theta_k} = 1 + \hat{d}_k\theta_k + \hat{d}_k^2\theta_k^2 + O(f^3(x_k)). \tag{2.40}$$

Using (2.39) and (2.40) in (2.22), we obtain

$$f(x_{k+1}) = O(f(x_k)^6). \tag{2.41}$$

This implies that the choice of the variable parameter (2.34) significantly accelerates the two-point method (2.1). The order of convergence increases from two to six. In this case, method (2.1) actually is a three-point one, i.e.,

$$\eta_k = x_k - \frac{f(x_k)}{f'(x_k)}, \quad y_k = x_k - \frac{f(x_k)}{\phi_k}, \quad x_{k+1} = y_k - \overline{\tau}_k\frac{f(y_k)}{\phi_k}. \tag{2.42}$$

If we replace $\gamma_k = -\frac{1}{f'(x_k)} \approx -\frac{1}{N_3'(x_k)}$, then we obtain a two-point iterative method with memory (x_0 and γ_0 are given). Then $\eta_0 = x_0 + \gamma_0 f(x_0)$ and

$$\begin{gathered}\gamma_k = -\frac{1}{N_3'(x_k)}, \quad \eta_k = x_k + \gamma_k f(x_k), \quad k = 1, 2, \ldots, \\ y_k = x_k - \frac{f(x_k)}{\phi_k}, \quad x_{k+1} = y_k - \overline{\tau}_k \frac{f(y_k)}{\phi_k}, \quad \phi_k = \phi(x_k, \gamma_k).\end{gathered} \tag{2.43}$$

Here, $N_3(t) = N_3(t, x_k, x_{k-1}, y_{k-1}, \eta_{k-1})$ is the Newton cubic interpolation polynomial specified by the node points x_k, x_{k-1}, y_{k-1}, and η_{k-1} [9], [23]. It is clear that the order R of methods (2.43) is at least six.

Note that sometimes asymmetric derivative-free iterations that require additional computations were used. For example, in [33] the optimal iterative families of the King type were proposed:

$$\begin{gathered}y_k = x_k - \frac{f(x_k)}{\phi_k}, \quad \phi_k = f[x_k, \eta_k], \quad \eta_k = x_k + \gamma f(x_k), \\ x_{k+1} = y_k - \frac{f(y_k)}{f[y_k, \eta_k]} \frac{1 + \beta\theta_k}{1 + (\beta - 1)\theta_k},\end{gathered} \tag{2.44}$$

where $f[x_k, \eta_k]$ is the first divided difference. The second substep in (2.44) can be written as

$$x_{k+1} = y_k - \overline{\tau}_k \frac{f(y_k)}{\phi_k}, \tag{2.45}$$

where

$$\overline{\tau}_k = \frac{1 + \beta\theta_k}{1 + (\beta - \hat{d}_k)\theta_k - \frac{(\beta - 1)\theta_k^2}{1 + \gamma\phi_k}}; \tag{2.46}$$

i.e., in this case $\overline{\tau}_n$ is determined by a more complicated formula than in (2.30). Moreover, as $\gamma \to 0$, we have

$$\overline{\tau}_k \to \frac{1 + \beta\theta_k}{1 + (\beta - 2)\theta_k - (\beta - 1)\theta_k^2},$$

while

$$\overline{\tau}_k \to \frac{1 + \beta\theta_k}{1 + (\beta - 2)\theta_k}$$

in (2.30). Another example of the finite difference version of the optimal Hansen–Patrick family proposed in [32] can be written as

$$\begin{gathered}y_k = x_k - \frac{f(x_k)}{\phi_k + \lambda f(\eta_k)}, \quad \eta_k = x_k + \gamma f(x_k), \quad \gamma, \lambda \in R\backslash\{0\}, \\ x_{k+1} = y_k - \frac{f(y_k)}{f[y_k, \eta_k] + \lambda f(\eta_k)} \overline{\tau}_k,\end{gathered} \tag{2.47}$$

where

$$\begin{gathered}\overline{\tau}_k = \frac{1}{\theta_k}\left(-1 + \frac{\alpha + 1}{\alpha + \sqrt{1 - 2(\alpha + 1)\theta_k}}\right) H(\theta_k), \quad \alpha \neq -1, \\ H(0) = 1, \quad H'(0) = -\frac{\alpha + 1}{2}, \quad |H''(0)| < \infty.\end{gathered} \tag{2.48}$$

Remove the asymmetry in (2.47) and consider the iterative method

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{\phi_k + \lambda f(\eta_k)}, \\ x_{k+1} &= y_k - \frac{f(y_k)}{\phi_k + \lambda f(\eta_k)}\overline{\tau}_k, \end{aligned} \tag{2.49}$$

where $\overline{\tau}_k$ as before is determined by formula (2.48). The following result is easy to prove.

Theorem 2. *Let $f(x) \in C^3(I)$ have a simple root $x^* \in I$. If the initial approximation x_0 is sufficiently close to $x^* \in I$, then the iterative method* (2.49) *has the optimal fourth order of convergence if*

$$H(0) = 1, \quad H'(0) = \hat{d}_k - \frac{\alpha+3}{2}, \quad |H''(0)| < \infty. \tag{2.50}$$

Proof. Assume that $H(0) = a$ and $H'(0) = b$. Then, we obtain from (2.48) that

$$\begin{aligned} \overline{\tau}_k &= \left(1 + \frac{\alpha+3}{2}\theta_k + \left(\frac{(\alpha+1)^2}{2} + \alpha + 2\right)\theta_k^2 + \cdots\right)(a + b\theta_k + O(f^2(x_k))) \\ &= a + \left(\frac{\alpha+3}{2}a + b\right)\theta_k + O(f^2(x_k)). \end{aligned}$$

By comparing this with the sufficient convergence condition (2.23), we conclude that

$$a = 1, \quad \frac{\alpha+3}{2} + b = \hat{d}_k \rightarrow b = \hat{d}_k - \frac{\alpha+3}{2}.$$

Therefore, by Theorem 1, the iterative method (2.49) has the fourth order of convergence under condition (2.50).

3. DERIVATIVE-FREE THREE-POINT ITERATIVE METHODS

Consider the derivative-free three-point methods

$$y_k = x_k - \frac{f(x_k)}{\phi(x_k)}, \quad z_k = y_k - \overline{\tau}_k\frac{f(y_k)}{\phi(x_k)}, \quad x_{k+1} = z_k - \alpha_k\frac{f(z_k)}{\phi(x_k)}, \tag{3.1}$$

which are obtained from the three-point methods studied in [6] by replacing $f'(x_k)$ with $\phi(x_k)$. Note that the first two steps in (3.1) determine optimal two-point fourth-order methods if $\overline{\tau}_k$ is given by (2.23). Our aim is to find α_k such that the order of convergence of iterations (3.1) is eight. To this end, we use the Taylor expansion of $f(x_{k+1})$:

$$\begin{aligned} f(x_{k+1}) &= f(z_k) - f'(z_k)\alpha_k\frac{f(z_k)}{\phi(x_k)} + O(f(z_k)^2) \\ &= \left(1 - \alpha_k\frac{f'(z_k)}{\phi(x_k)}\right)f(z_k) + O(f^2(z_k)). \end{aligned} \tag{3.2}$$

This implies that

$$f(x_{k+1}) = O(f^8(x_k)) \tag{3.3}$$

under the condition

$$\alpha_k = \frac{\phi(x_k)}{f'(z_k)} + O(f^4(x_k)). \tag{3.4}$$

Now we approximate $f'(z_k)$ in (3.4) using $f(x_k)$, $f(y_k)$, $f(z_k)$, and $\phi(x_k)$ such that

$$f'(z_k) = a_kf(x_k) + b_kf(y_k) + c_kf(z_k) + d_k\phi(x_k) + O(f(x_k)^4). \tag{3.5}$$

Using the Taylor expansion of $f(x)$ about the point z_k, we obtain the system of equations

$$\begin{aligned} a_k + b_k + c_k &= 0, \\ a_k w_k + b_k \gamma_k + d_k &= 1, \\ a_k w_k^2 + b_k \gamma_k^2 + 2d_k\left(w_k + \frac{1}{2}\gamma f(x_k)\right) &= 0, \\ a_k w_k^3 + b_k \gamma_k^3 + 3d_k\left(w_k^2 + w_k \gamma f(x_k) + \frac{1}{3}\gamma^2 f^2(x_k)\right) &= 0, \end{aligned} \tag{3.6}$$

where

$$w_k = x_k - z_k, \quad \gamma_k = y_k - z_k. \tag{3.7}$$

System (3.6) has the unique solution

$$\begin{aligned} c_k &= -a_k - b_k, \\ d_k &= 1 - a_k w_k - b_k \gamma_k, \\ b_k &= \frac{w_k(w_k + \gamma f(x_k))}{\gamma_k(\gamma_k - w_k)(\gamma_k - w_k - \gamma f(x_k))}, \\ a_k &= \frac{\gamma_k}{w_k(\gamma_k - w_k)} \frac{(w_k - \gamma_k)(2w_k + \gamma f(x_k)) + (w_k + \gamma f(x_k))^2}{(w_k + \gamma f(x_k))(w_k - \gamma_k + \gamma f(x_k))}. \end{aligned} \tag{3.8}$$

Substitute (3.8) into (3.5) to obtain

$$f'(z_k) = \phi_k(1 + a_k w_k\left(\frac{f[z_k, x_k]}{\phi_k} - 1\right) + b_k \gamma_k\left(\frac{f[z_k, y_k]}{\phi_k} - 1\right)) + O(f^4(x_k)), \tag{3.9}$$

where

$$\phi_k = \phi(x_k) = f[x_k, \xi_k], \quad \xi_k = x_k + \gamma f(x_k).$$

According to (3.2) and (3.7), we have

$$w_k = \frac{f(x_k)}{\phi_k}\tau_k, \quad \gamma_k = (\tau_k - 1)\frac{f(x_k)}{\phi_k}, \quad \gamma_k - w_k = -\frac{f(x_k)}{\phi_k} = y_k - x_k, \tag{3.10}$$

$$\frac{\gamma_k}{w_k - \gamma_k} = \frac{y_k - z_k}{x_k - y_k} = \tau_k - 1 \to \tau_k = \frac{x_k - z_k}{x_k - y_k}, \tag{3.11}$$

$$w_k + \gamma f_k = \frac{f(x_k)}{\phi_k}(\tau_k + \gamma\phi_k), \quad w_k - \gamma_k + \gamma f(x_k) = \frac{f(x_k)}{\phi_k}(1 + \gamma\phi_k). \tag{3.12}$$

Using (3.10)–(3.12) in (3.8), we conclude that

$$b_k \gamma_k = \frac{\tau_k(\tau_k + \gamma\phi_k)}{1 + \gamma\phi_k}, \quad a_k w_k = (1 - \tau_k)\frac{2\tau_k + \gamma\phi_k + (\tau_k + \gamma\phi_k)^2}{(\tau_k + \gamma\phi_k)(1 + \gamma\phi_k)}. \tag{3.13}$$

Substitute (3.9) into (3.4) and neglect the small term $O(f^4(x_k))$ to find that

$$\alpha_k = \frac{1}{1 + a_k w_k\left(\frac{f[z_k, x_k]}{\phi_k} - 1\right) + b_k \gamma_k\left(\frac{f[z_k, y_k]}{\phi_k} - 1\right)}, \tag{3.14}$$

where $a_k w_k$ and $b_k \gamma_k$ are determined by formula (3.13). The expressions in parentheses in (3.14) can be rewritten in terms of the second divided differences as

$$\frac{f[z_k, x_k]}{\phi_k} - 1 = \frac{1}{\phi_k} f[z_k, x_k, \xi_k](z_k - \xi_k) = -\frac{f(x_k)}{\phi_k^2} f[z_k, x_k, \xi_k](\tau_k + \gamma\phi_k), \tag{3.15}$$

$$\frac{f[z_k, y_k]}{\phi_k} - 1 = -\frac{f(x_k)}{\phi_k^2}(f[y_k, z_k, x_k] + f[z_k, x_k, \xi_k](\tau_k + \gamma\phi_k)). \tag{3.16}$$

By substituting (3.13), (3.15), and (3.16) into (3.14), we obtain another representation of α_k:

$$\alpha_k = \frac{1}{1 - \dfrac{f(x_k)}{\phi_k^2(1+\gamma\phi_k)} F_k}; \tag{3.17}$$

here $F_k = (\tau_k(\tau_k + \gamma\phi_k) f[y_k, z_k, x_k] + ((1-\tau_k)(2\tau_k + \gamma\phi_k) + (\tau_k + \gamma\phi_k)^2) f[z_k, x_k, \xi_k])$.

Now, we are going to find an asymptotic formula for α_k defined by (3.14). To this end, we use the formulas

$$\frac{f[z_k, x_k]}{\phi_k} - 1 = -\frac{(\overline{\tau}_k + v_k)\theta_k}{1 + \overline{\tau}_k\theta_k}, \tag{3.18}$$

$$\frac{f[z_k, y_k]}{\phi_k} - 1 = \frac{1 - \overline{\tau}_k - v_k}{\overline{\tau}_k}, \quad v_k = f(z_k)/f(y_k), \tag{3.19}$$

$$\tau_k = 1 + \overline{\tau}_k\theta_k. \tag{3.20}$$

Similarly, (3.13) can be rewritten in terms $\overline{\tau}_k$ as (3.18) and (3.19). Taking this into account and using (3.18) and (3.19), we can rewrite (3.14) as

$$\alpha_k = \frac{1}{1 + \dfrac{A_1 + A_2 + A_3 v_k}{(1+\gamma\phi_k)(1+\gamma\phi_k + \overline{\tau}_k\theta_k)(1+\overline{\tau}_k\theta_k)\overline{\tau}_k}}, \tag{3.21}$$

where

$$A_1 = (1 + \theta_k\overline{\tau}_k)^2(1 + \gamma\phi_k + \overline{\tau}_k\theta_k)^2(1 - \overline{\tau}_k), \tag{3.22a}$$

$$A_2 = (2 + \gamma\phi_k + 2\overline{\tau}_k\theta_k + (1 + \gamma\phi_k + \overline{\tau}_k\theta_k)^2)\overline{\tau}_k^3\theta_k^2, \tag{3.22b}$$

$$\begin{aligned} A_3 = &-(1 + \theta_k\overline{\tau}_k)^2(1 + \gamma\phi_k + \overline{\tau}_k\theta_k)^2 \\ &+ (2 + \gamma\phi_k + 2\overline{\tau}_k\theta_k + (1 + \gamma\phi_k + \overline{\tau}_k\theta_k)^2)\overline{\tau}_k^2\theta_k^2. \end{aligned} \tag{3.22c}$$

Due to (2.23), we may write $\overline{\tau}_k$ as

$$\overline{\tau}_k = 1 + \hat{d}_k\theta_k + \tilde{\beta}_k\theta_k^2 + \tilde{\gamma}_k\theta_k^3 + \cdots, \tag{3.23}$$

where $\tilde{\beta}_k$ and $\tilde{\gamma}_k$ are constants. Then, by Theorem 1 we have

$$f(y_k) = O(f^2(x_k)), \quad f(z_k) = O(f^4(x_k)), \quad v_k = O(f^2(x_k)). \tag{3.24}$$

Using (3.23) and (3.24) in (3.22), we obtain

$$A_1 = -\theta_k(1 + \gamma\phi_k)^2(a_1 + a_2\theta_k + a_3\theta_k^2 + \cdots), \tag{3.25}$$

$$A_2 = \theta_k^2(1 + \gamma\phi_k)^2(b_1 + b_2\theta_k + \cdots), \tag{3.26}$$

$$A_3 = -(1 + \gamma\phi_k)^2(c_1 + c_2\theta_k + \cdots), \tag{3.27}$$

where

$$\begin{aligned} &a_1 = \hat{d}_k, \quad a_2 = \tilde{\beta} + 2\hat{d}_k^2, \quad a_3 = \tilde{\gamma} + 2\tilde{\beta}\hat{d}_k + \left(3\hat{d}_k^2 + \frac{2}{1+\gamma\phi_k}\right)\hat{d}_k, \\ &b_1 = 1 + \frac{\hat{d}_k}{1+\gamma\phi_k}, \quad b_2 = \hat{d}_k - \hat{d}_k^2 + 3\hat{d}_k^3, \quad c_1 = 1, \quad c_2 = 2\hat{d}_k. \end{aligned} \tag{3.28}$$

Similarly, we have

$$\frac{1}{\left(1 + \dfrac{\overline{\tau}_k\theta_k}{1+\gamma\phi_k}\right)(1 + \overline{\tau}_k\theta_k)\overline{\tau}_k} = 1 - 2\hat{d}_k\theta_k + \left(2\hat{d}_k^2 - \tilde{\beta} - \frac{1}{1+\gamma\phi_k}\right)\theta_k^2 + O(f^3(x_k)). \tag{3.29}$$

Then

$$\frac{A_1 + A_2 + A_3 v_k}{(1+\gamma\phi_k)(1+\gamma\phi_k+\bar{\tau}_k\theta_k)(1+\bar{\tau}_k\theta_k)\bar{\tau}_k} = \left(1 - 2\hat{d}_k\theta_k + \left(2\hat{d}_k^2 - \tilde{\beta} - \frac{1}{1+\gamma\phi_k}\right)\theta_k^2\right)$$
$$\times(-a_1\theta_k + (b_1 - a_2)\theta_k^2 + (b_2 - a_3)\theta_k^3 - (c_1 + c_2\theta_k)v_k)$$
$$= -a_1\theta_k + (b_1 - a_2 + 2a_1\hat{d}_k)\theta_k^2 + \Big(b_2 - a_3 - 2\hat{d}_k(b_1 - a_2)$$
$$- a_1\left(2\hat{d}_k^2 - \tilde{\beta} - \frac{1}{1+\gamma\phi_k}\right)\Big)\theta_k^3 - (c_1 + (c_2 - 2c_1\hat{d}_k)\theta_k)v_k + O(f^4(x_k)).$$

Substitute this expression into (3.21) and use the known expansion (2.20) to obtain

$$\alpha_k = 1 + a_1\theta_k - (b_1 - a_2 + 2a_1\hat{d}_k)\theta_k^2 + (2\hat{d}_k(b_1 - a_2) + a_1\left(2\hat{d}_k^2 - \tilde{\beta} - \frac{1}{1+\gamma\phi_k}\right) - (b_2 - a_3))\theta_k^3$$
$$+ (c_1 + (c_2 - 2c_1\hat{d}_k)\theta_k)v_k + a_1^2\theta_k^2 - 2a_1(b_1 - a_2 + 2a_1\hat{d}_k)\theta_k^3 + 2a_1c_1\theta_k v_k + a_1^3\theta_k^3 + O(f^4(x_k))$$
$$= 1 + a_1\theta_k + \left(a_1^2 - (b_1 - a_2) - 2a_1\hat{d}_k\right)\theta_k^2 + \Big((a_1^3 + 2\hat{d}_k(b_1 - b_2) + a_1\left(2\hat{d}_k^2 - \tilde{\beta} - \frac{1}{1+\gamma\phi_k}\right)\Big)$$
$$- (b_2 - a_3) - 2a_1(b_1 - a_2 + 2a_1\hat{d}_k))\theta_k^3 + (c_1 + (c_2 - 2c_1\hat{d}_k)\theta_k)v_k + O(f^4(x_k))$$

or

$$\alpha_k = 1 + \hat{d}_k\theta_k + \left(\tilde{\beta} + \frac{1}{1+\gamma\phi_k}\right)\theta_k^2 + \left(\tilde{\gamma} + \hat{d}_k\tilde{\beta} - \hat{d}_k - \frac{\hat{d}_k}{(1+\gamma\phi_k)^2}\right)\theta_k^3 + (1 + 2\hat{d}_k\theta_k)v_k + O(f^4(x_k)). \quad (3.30)$$

As $\gamma \to 0$, formula (3.30) reduces to the form

$$\alpha_k = 1 + 2\theta_k + (\tilde{\beta} + 1)\theta_k^2 + (\tilde{\gamma} + 2\tilde{\beta} - 4)\theta_k^3 + (1 + 4\theta_k)v_k + O(f^4(x_k)), \quad (3.31)$$

which describes the asymptotic behavior of α_k in three-point iterative methods (see [6]).

Theorem 3. *Let all assumptions of Theorem* 1 *be fulfilled. Then, the three-point iterative methods* (3.1) *have the eighth order of convergence if and only if the iteration parameters* $\bar{\tau}_k$ *and* α_k *are specified by formulas* (3.23) *and* (3.30)*, respectively.*

Proof. Let $\bar{\tau}_k$ and α_k be defined by formulas (3.23) and (3.30), respectively. Then, by Theorem 1 the first two steps in (3.1) determine an optimal fourth-order method, i.e., $f(z_k) = O(f^4(x_k))$. The value α_k specified by formula (3.30) satisfies condition (3.4). Therefore, we have (3.3). Conversely, assume that the order of convergence of (3.1) is eight. Then (3.1) and (3.3) imply that $f(z_k) = O(f^4(x_k))$ and formula (3.4) is valid. Therefore, by Theorem 1 formula (3.23) is valid for certain constants $\tilde{\gamma}$ and $\tilde{\beta}$. Using approximation (3.9) in (3.4), we obtain (3.14) accurate to $O(f^4(x_k))$. Due to (3.23), we obtain from (3.14) the asymptotic formula (3.30).

Assume that in (3.1)

$$\bar{\tau}_k = H(\theta_k) = \frac{c + (\hat{d}_k c + d)\theta_k + \omega\theta_k^2}{c + d\theta_k + b\theta_k^2}, \quad c + d + b \neq 0, \quad c, d, b, \omega \in R, \quad (3.32)$$

$$\alpha_k = H(\theta_k) + \frac{1}{1+\gamma\phi_k}\theta_k^2 + \hat{d}_k\left(\tilde{\beta} - \frac{2}{1+\gamma\phi_k}\right)\theta_k^3 + (1 + 2\hat{d}_k\theta_k)v_k. \quad (3.33)$$

Then, we obtain a family of optimal derivative-free three-point iterative methods because $\bar{\tau}_k$ and α_k determined by (2.23) and (3.33) satisfy conditions (3.23) and (3.30) with the constants

$$\tilde{\beta} = \frac{\omega - b}{c} - \frac{d}{c}\left(\frac{d}{c} + \hat{d}_k\right), \quad \tilde{\gamma} = -\frac{(b+\omega)d}{c^2} + \frac{d^2 - bc}{c^2}\hat{d}_k,$$

respectively. Therefore, the generation function method described in [5] makes it possible to construct the family of optimal three-point iterations.

Table 2. Nonlinear functions

	Functions	Root
1.	$f_1(x) = e^{(x^2+x\cos x-1)}\sin x + x\log(x\sin x + 1)$, [9]	$x^* = 0$
2.	$f_2(x) = \log(x^2 - 2x + 2) + e^{(x^2-5x+4)}\sin(x-1)$,	$x^* = 1$
3.	$f_3(x) = \begin{cases} x(x+1) & \text{if } x < 0, \\ -2x(x-1) & \text{if } x \geq 0, \end{cases}$ [7, 32]	$x^* = 0$ $x^* = 1$ $x^* = -1$
4.	$f_4(x) = \lvert x^2 - 4\rvert$, [9]	$x^* = \pm 2$

Now consider the three-point iterative method

$$\begin{gathered} \eta_k = x_k + \gamma f(x_k), \quad y_k = x_k - \frac{f(x_k)}{f[x_k, \eta_k]}, \quad z_k = \psi_4(x_k, y_k, \eta_k), \\ x_{k+1} = z_k - \frac{f(z_k)}{f[z_k, y_k] + (z_k - y_k)f[z_k, y_k, x_k] + (z_k - y_k)(z_k - x_k)f[z_k, y_k, x_k, \eta_k]}. \end{gathered} \tag{3.34}$$

Here the function ψ_4 is taken from any optimal derivative-free fourth-order method and $f[z_k, y_k, x_k, \eta_k]$ is the third divided difference. Theorem 2 implies the following result.

Theorem 4. *Let all assumptions of Theorem* 1 *be fulfilled. Then, the order of convergence of the iterative method* (3.34) *is eight.*

Proof. Since ψ_4 is a fourth-order iteration, z_k can be rewritten as

$$z_k = y_k - \bar{\tau}_k \frac{f(y_k)}{\phi(x_k)}, \quad \phi(x_k) = f[x_k, \eta_k].$$

By Theorem 1, we have $\bar{\tau}_k = 1 + \hat{d}_k\theta_k + O(f^2(x_k))$. This implies the Taylor expansion (3.23) for $\bar{\tau}_k$. By comparing (3.1) with (3.34), we obtain

$$\alpha_k = \frac{\phi_k}{f[z_k, y_k] + (z_k - y_k)f[z_k, y_k, x_k] + (z_k - y_k)(z_k - x_k)f[z_k, y_k, x_k, \eta_k]} \tag{3.35}$$

It is easy to verify that the parameter α_k defined by formula (3.35) satisfies condition (3.30). Then, Theorem 3 implies that the order of convergence of (3.34) is eight 8.

Remark 1. The order of convergence of the three-point iterative methods proposed in [12, 22–24] immediately follows from Theorem 4, which is an extension of the theorems in [12, 22–24].

Note that all existing optimal derivative-free methods can be unambiguously written in form (3.1).

It is easy to verify that the parameters $\bar{\tau}_k$ and α_k in these methods have the same asymptotics (3.23) and (3.30) with specific constants $\tilde{\gamma}$ and $\tilde{\beta}$. Thus, the convergence of all existing optimal derivative-free methods can be proved using the sufficient convergence conditions (3.23) and (3.30) without symbolic computations. Furthermore, the application of these sufficient convergence conditions makes it possible to construct new optimal iterative methods [29]. It is seen from Table 1 that the parameter $\bar{\tau}_k$ in all optimal three-point methods listed in it is obtained using the generating functions $H(\theta_k)$ determined by (3.32); the only exception is the method proposed in [16]. It is seen from (3.32) and (3.34) that the function ψ_4 can contain free parameters. This implies that the iterative methods (3.34) form a wide class of optimal derivative-free three-point methods. This class includes many well-known methods as special cases (see [4–6, 9, 12]). As in the preceding section, we can vary γ at each iteration step using the information

Table 3. Two-point iterative methods

Methods	$\overline{\tau}_k$	k	$\lvert x^* - x_k\rvert$	COC
Numerical results for the smooth function $f_1(x)$ with $x_0 = 1$				
(2.1)	$c = 1, d = -\hat{d}_k, b = -\dfrac{1}{1+\gamma\varphi_k}, \omega = 0$	4	0.4180e–33	3.99
King-type [23]	$c = 1, d = -\hat{d}_k, b = \dfrac{1}{1+\gamma\varphi_k}, \omega = 0$	5	0.5272e–96	4.00
Potra–Ptak's [13, 22]	$c = 1, d = b = 0, \omega = \dfrac{\hat{d}_k}{2}$	5	0.9744e–80	3.99
$P1$ [21]	$c = 1, b = d = \omega = 0$	5	0.1887e–65	4.00
$P2$ [21]	$c = 1, d = -\dfrac{1}{1+\gamma\varphi_k}, b = \omega = 0$	5	0.1022e–95	4.00
Zheng's [12]	$c = 1, d = -\hat{d}_k, b = \omega = 0$	4	0.1655e–35	4.00
(2.31)	$c = b = 1, d = -2, \omega = 0$	5	0.1416e–95	4.00
(2.32)	$c = 1, d = \omega = -1, b = 0$	5	0.3838e–82	3.99
Steffensen's	$x_{k+1} = x_k - \dfrac{f(x_k)}{\phi(x_k)}$	9	0.8745e–58	2.00
Numerical results for the smooth function $f_2(x)$ with $x_0 = 0.5$				
(2.1)	$c = 1, d = -\hat{d}_k, b = -\dfrac{1}{1+\gamma\varphi_k}, \omega = 0$	4	0.1673e–104	4.00
King-type [23]	$c = 1, d = -\hat{d}_k, b = \dfrac{1}{1+\gamma\varphi_k}, \omega = 0$	5	0.8607e–112	4.00
Potra–Ptak's [13, 22]	$c = 1, d = b = 0, \omega = \dfrac{\hat{d}_k}{2}$	5	0.4066e–70	4.00
$P1$ [21]	$c = 1, b = d = \omega = 0$	5	0.1325e–62	4.00
$P2$ [21]	$c = 1, d = -\dfrac{1}{1+\gamma\varphi_k}, b = \omega = 0$	5	0.5680e–88	4.00
Zheng's [12]	$c = 1, d = -\hat{d}_k, b = \omega = 0$	4	0.4934e–58	3.99
(2.31)	$c = b = 1, d = -2, \omega = 0$	5	0.6144e–109	4.00
(2.32)	$c = 1, d = \omega = -1, b = 0$	5	0.6129e–73	4.00
Steffensen's	$x_{k+1} = x_k - \dfrac{f(x_k)}{\phi(x_k)}$	8	0.4282e–30	2.00

obtained at the preceding and the current steps. This enables us to increase the order of convergence without using additional computations. More precisely, we can obtain three-point iterative methods with memory (x_0 and γ_0 are given). Then, $\eta_0 = x_0 + \gamma_0 f(x_0)$ and

$$\gamma_k = -\frac{1}{N_4'(x_k)}, \quad \eta_k = x_k + \gamma_k f(x_k), \quad k = 1, 2, \ldots$$
$$y_k = x_k - \frac{f(x_k)}{\phi(x_k, \gamma_k)}, \quad z_k = \psi_4(x_k, y_k, \eta_k), \tag{3.36}$$
$$x_{k+1} = z_k - \frac{f(z_k)}{f[z_k, y_k] + (z_k - y_k)f[z_k, y_k, x_k] + (z_k - y_k)(z_k - x_k)f[z_k, y_k, x_k, \eta_k]}.$$

Table 4. Three-point iterative methods

Methods	$\overline{\tau}_k = H(\theta_k)$ / choice of parameters	k	$\lvert x^* - x_k\rvert$	COC
	Numerical results for the smooth function $f_1(x)$ with $x_0 = 1$			
(3.34)	$c = 1, d = \beta - 2, b = \omega = 0, (\beta = 2)$	3	0.1710e–38	8.38
(3.34)	$c = b = 1, d = -2, \omega = 0$	3	0.3900e–57	7.94
(3.34)	$c = 1, d = \omega = -1, b = 0$	3	0.4900e–44	7.99
Lotfi's [22]	$c = 1, d = b = 0, \omega = \frac{\tilde{d}_k}{2}$	3	0.4362e–43	7.99
King-type [23]	$c = \omega = 1, d = \beta - 1 - \tilde{d}_k, b = \frac{2-\beta}{1+\gamma\varphi_k}, (\beta = 2)$	3	0.1024e–54	7.98
Zheng's [12]	$c = 1, d = -\hat{d}_k, b = \omega = 0$	3	0.5610e–62	7.97
Sharma's [14]	$c = 1, d = -\frac{1}{1+\gamma\varphi_k}, b = \omega = 0$	3	0.9068e–48	8.00
	Numerical results for the smooth function $f_2(x)$ with $x_0 = 0.5$			
(3.34)	$c = 1, d = \beta - 2, b = \omega = 0, (\beta = 2)$	3	0.3321e–33	7.96
(3.34)	$c = b = 1, d = -2, \omega = 0$	3	0.1543e–44	8.07
(3.34)	$c = 1, d = \omega = -1, b = 0$	3	0.4989e–36	7.98
Lotfi's [22]	$c = 1, d = b = 0, \omega = \frac{\tilde{d}_k}{2}$	3	0.2769e–35	7.99
King-type [23]	$c = \omega = 1, d = \beta - 1 - \tilde{d}_k, b = \frac{2-\beta}{1+\gamma\varphi_k}, (\beta = 2)$	3	0.5302e–44	8.00
Zheng's [12]	$c = 1, d = -\hat{d}_k, b = \omega = 0$	3	0.6281e–64	7.97
Sharma's [14]	$c = 1, d = -\frac{1}{1+\gamma\varphi_k}, b = \omega = 0$	3	0.7441e–40	8.02

Here $N_4(t) = N_4(t, x_k, z_{k-1}, y_{k-1}, \eta_{k-1}, x_{k-1})$ the fourth degree interpolation Newton polynomial specified by the node points x_k, z_{k-1}, y_{k-1}, η_{k-1}, x_{k-1}. As in [9], it is easy to prove that the order R of convergence of method (3.36) is at least 12.

4. NUMERICAL RESULTS

In this section, we describe the results of numerical computations for comparing the effectiveness of different methods. The computations were performed in Maple. To ensure high accuracy and avoid losing significant digits, the computations were performed with 300 significant digits. The computations were performed for smooth and nonsmooth functions (see Table 2) with $\gamma = -0.01$. To check the convergence of Newtons, the computational order of convergence (COC) was calculated by the formula

$$p \approx \frac{\ln(|x_k - x^*|/|x_{k-1} - x^*|)}{\ln(|x_{k-1} - x^*|/|x_{k-2} - x^*|)},$$

Table 5. Numerical results for the nonsmooth function $f_3(x)$. Three-point iterative methods

Methods	$\overline{\tau}_k = H(\theta_k)$	k	$\lvert x^* - x_k \rvert$	COC
	$x_0 = 0.1$, $x^* = 0$			
(3.34)	$c = 1, d = \beta - 2, b = \omega = 0, (\beta = 2)$	4	0.7235e–30	2.00
(3.34)	$c = b = 1, d = -2, \omega = 0$	4	0.7186e–30	2.00
(3.34)	$c = 1, d = \omega = -1, b = 0$	4	0.7222e–30	2.00
Lotfi's [22]	$c = 1, d = b = 0, \omega = \frac{\tilde{d}_k}{2}$	4	0.7221e–30	2.00
King-type [23]	$c = \omega = 1, d = \beta - 1 - \tilde{d}_k, b = \frac{2-\beta}{1+\gamma\varphi_k}, (\beta = 2)$	4	0.7185e–30	2.00
Zheng's [12]	$c = 1, d = -\hat{d}_k, b = \omega = 0$	4	0.7167e–30	2.00
Sharma's [14]	$c = 1, d = -\frac{1}{1+\gamma\varphi_k}, b = \omega = 0$	4	0.7205e–30	2.00
	$x_0 = 5$, $x^* = 1$			
(3.34)	$c = 1, d = \beta - 2, b = \omega = 0, (\beta = 2)$	4	0.2191e–236	7.99
(3.34)	$c = b = 1, d = -2, \omega = 0$	3	0.8113e–39	7.77
(3.34)	$c = 1, d = \omega = -1, b = 0$	3	0.8754e–32	7.60
Lotfi's [22]	$c = 1, d = b = 0, \omega = \frac{\tilde{d}_k}{2}$	3	0.2144e–31	7.60
King-type [23]	$c = \omega = 1, d = \beta - 1 - \tilde{d}_k, b = \frac{2-\beta}{1+\gamma\varphi_k}, (\beta = 2)$	3	0.2249e–37	7.76
Zheng's [12]	$c = 1, d = -\hat{d}_k, b = \omega = 0$	3	0.5377e–47	7.86
Sharma's [14]	$c = 1, d = -\frac{1}{1+\gamma\varphi_k}, b = \omega = 0$	3	0.4975e–34	7.67
	$x_0 = -10$, $x^* = -1$			
(3.34)	$c = 1, d = \beta - 2, b = \omega = 0, (\beta = 2)$	4	0.4791e–102	7.99
(3.34)	$c = b = 1, d = -2, \omega = 0$	4	0.2067e–141	7.99
(3.34)	$c = 1, d = \omega = -1, b = 0$	4	0.9351e–112	7.99
Lotfi's [22]	$c = 1, d = b = 0, \omega = \frac{\tilde{d}_k}{2}$	4	0.5302e–110	7.99
King-type [23]	$c = \omega = 1, d = \beta - 1 - \tilde{d}_k, b = \frac{2-\beta}{1+\gamma\varphi_k}, (\beta = 2)$	4	0.2101e–135	7.99
Zheng's [12]	$c = 1, d = -\hat{d}_k, b = \omega = 0$	4	0.8976e–178	7.99
Sharma's [14]	$c = 1, d = -\frac{1}{1+\gamma\varphi_k}, b = \omega = 0$	4	0.2099e–121	7.99

where x_k, x_{k-1}, and x_{k-2} are three successive approximations. The iterative process is stopped when $|x_k - x^*| < 10^{-30}$.

Table 2 presents the example taken from [9]. The third example with the nonsmooth function (see [7, 14, 32]) is often used for checking the validity of derivative-free iterative methods. Tables 3–6 show the number of iteration steps (k), the absolute errors $|x_k - x^*|$, and COC for the methods with $\gamma = -0.01$. The numerical results confirm the theoretical conclusion about the order of convergence. It is seen from Table 6 that the high order methods work well not only for sufficiently smooth functions but also for nonsmooth ones. Note that the derivative of the nonlinear function $f_3(x)$ has a discontinuity at the point $x^* = 0$; for

Table 6. Three-point iterative methods

Methods	$\overline{\tau}_k = H(\theta_k)$ choice of parameters	k	$\lvert x^* - x_k\rvert$	COC
Numerical results for the nonsmooth function $f_4(x)$ with $x_0 = 3$				
(3.34)	$c = 1, d = \beta - 2, b = \omega = 0, (\beta = 2)$	2	0.1365e–35	7.70
(3.34)	$c = b = 1, d = -2, \omega = 0$	2	0.3071e–40	7.79
(3.34)	$c = 1, d = \omega = -1, b = 0$	2	0.8144e–37	7.72
Lotfi's [22]	$c = 1, d = b = 0, \omega = \frac{\tilde{d}_k}{2}$	2	0.1228e–36	7.72
King-type [23]	$c = \omega = 1, d = \beta - 1 - \tilde{d}_k, b = \frac{2-\beta}{1+\gamma\varphi_k}, (\beta = 2)$	2	0.3717e–40	7.80
Zheng's [12]	$c = 1, d = -\hat{d}_k, b = \omega = 0$	2	0.1675e–44	7.84
Sharma's [14]	$c = 1, d = -\frac{1}{1+\gamma\varphi_k}, b = \omega = 0$	2	0.2114e–38	7.75
Steffensen's	$x_{k+1} = x_k - \frac{f(x_k)}{\phi(x_k)}$	6	0.5556e–45	2.00

this reason, the COC = 2 in this case for all the methods discussed in this paper (see the first part of Table 5 and [7]). The proposed methods (3.34) can be successfully used in the computations that require high accuracy.

5. CONCLUSIONS

The necessary and sufficient convergence conditions for two- and three-point iterative methods obtained in [6] are extended for the case of derivative-free methods. The latter methods can be effectively used not only for determining the order of convergence but also for constructing new methods. Based on the generating function method, wide classes of optimal methods that include many known methods as special cases are proposed.

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REFERENCES

1. L. Liu and X. Wang, "Eighth-order methods of high efficiency index for solving nonlinear equations," Appl. Math. Comput. **215**, 3449–3454 (2010).
2. M. S. Petković, B. Neta, L. D. Petković, and J. Džunić, "Multipoint methods for solving nonlinear equations: A survey," Appl. Math. Comput. **226**, 635–660 (2014).
3. R. Thukral and M. S. Petković, "A family of three-point methods of optimal order for solving nonlinear equations," J. Comput. Appl. Math. **233**, 2278–2284 (2010).
4. X. Wang and L. Liu, "New eighth-order iterative methods for solving non-linear equations," J. Comput. Appl. Math. **234**, 1611–1620 (2010).
5. T. Zhanlav, V. Ulziibayar, O. Chuluunbaatar, and "Generating function method for constructing new iterations," Appl. Math. Comput. **315**, 414–423 (2017).
6. T. Zhanlav, V. Ulziibayar, and O. Chuluunbaatar, "Necessary and sufficient conditions for the convergence of two- and three-point Newton-type iterations," Comput. Math. Math. Phys. **57**, 1090–1100 (2017).
7. A. Cordero, J. L. Hueso, E. Martinez, and J. R. Torregrosa, "A new technique to obtain derivative-free optimal iterative methods for solving nonlinear equations," J. Comput. Appl. Math. **252**, 95–102 (2013).
8. M. D. Junjua, F. Zafar, N. Yasmin, and S. Akram, "A general class of derivative free with memory root solvers," Appl. Math. Phys. **79**, 19–28 (2017).

9. I. K. Argyros, M. Kansal, V. Kanwar, and S. Bajaj, "Higher-order derivative-free families of Chebyshev—Halley type methods with or without memory for solving nonlinear equations," Appl. Math. Comput. **315**, 224–245 (2017).

10. S. K. Khattri and T. Steihaug, "Algorithm for forming derivative-free optimal methods," Numer. Algorithms **5**, 809–842 (2014).

11. R. Thukral, "Eighth-order iterative methods without derivatives for solving nonlinear equations," ISRN Appl. Math. Article ID 693787 (2011).

12. Q. Zheng, J. Li, and F. Huang, "An optimal Steffensen-type family for solving nonlinear equations," Appl. Math. Comput. **217**, 9592–9597 (2011).

13. F. Soleymani, R. Sharma, X. Li, and E. Tohidi, "An optimized derivative-free form of the Potra-Ptak method," Math. Comput. Model. **56**, 97–104 (2012).

14. J. R. Sharma and R. K. Goyal, "Fourth-order derivative-free methods for solving non-linear equations," Int. J. Comput. Math. **83**, 101–106 (2006).

15. A. Cordero and J. R. Torregrosa, "A class of Steffensen type methods with optimal order of convergence," Appl. Math. Comput. **217**, 7653–7659 (2011).

16. H. Ren, Q. Wu, and W. Bi, "A class of two-step Steffensen type methods with fourth-order convergence," Appl. Math. Comput. **209**, 206–210 (2009).

17. Z. Liu, Q. Zheng, and P. Zhao, "A variant of Steffensen's method of fourth-order convergence and its applications," Appl. Math. Comput. **216**, 1978–1983 (2010).

18. Y. Peng, H. Feng, Q. Li, and X. Zhang, "A fourth-order derivative-free algorithm for nonlinear equations," J. Comput. Appl. Math. **235**, 2551–2559 (2011).

19. M. Kansal, V. Kanwar, and S. Bhatia, "An optimal eighth-order derivative-free family of Potra-Ptak's method," Algorithms **8**, 309–320 (2015).

20. F. Soleymani and S. K. Khattri, "Finding simple roots by seventh-and eighth-order derivative-free methods," Int. J. Math. Models Meth. Appl. Sci. **6**, 45–52 (2012).

21. R. Thukral, "A family of three-point derivative-free methods of eighth-order for solving nonlinear equations," J. Mod. Meth. Numer. Math. **3**, 11–21 (2012).

22. T. Lotfi, F. Soleymani, M. Ghorbanzadeh, and P. Assari, "On the construction of some tri-parametric iterative methods with memory," Numer. Algorithms **70**, 835–845 (2015).

23. S. Sharifi, S. Siegmund, and M. Salimi, "Solving nonlinear equations by a derivative-free form of the King's family with memory," Calcolo **53**, 201–215 (2016).

24. J. R. Sharma, R. K. Guha, and P. Gupta, "Some efficient derivative free methods with memory for solving nonlinear equations," Appl. Math. Comput. **219**, 699–707 (2012).

25. R. Behl, D. Gonzalez, P. Maroju, and S. S. Motsa, "An optimal and efficient general eighth-order derivative-free scheme for simple roots," J. Comput. Appl. Math. **330**, 666–675 (2018).

26. F. Soleymani, "Optimal fourth-order iterative methods free from derivative," Miskolc Math. Notes. **12**, 255–264 (2011).

27. S. K. Khattri and R. P. Agarwal, "Derivative-free optimal iterative methods," Comput. Meth. Appl. Math. **10**, 368–375 (2010).

28. T. Zhanlav and Kh. Otgondorj, "A new family of optimal eighth-order methods for solving nonlinear equations," Am. J. Comput. Appl. Math. **8**, 15–19 (2018).

29. F. Zafar, N. Yasmin, M. A. Kutbi, and M. Zeshan, "Construction of Tri-parametric derivative free fourth order with and without memory iterative method," J. Nonlinear Sci. Appl. **9**, 1410–1423 (2016).

30. F. Soleymani and S. K. Vanani, "Optimal Steffensen-type methods with eighth order of convergence," Comput. Math. Appl. **62**, 4619–4626 (2011).

31. F. Soleymani, "On a bi-parametric class of optimal eighth-order derivative-free methods," Int. J. Pure. Appl. Math. **72**, 27–37 (2011).

32. M. Kansal, V. Kanwar, and S. Bhatia, "Efficient derivative-free variants of Hansen-Patrick's family with memory for solving nonlinear equations," Numer. Algor. **73**, 1017–1036 (2016).

33. F. I. Chicharro, A. Cordero, J. R. Torregrosa, and M. P. Vassileva, "King-Type derivative-free iterative families: Real and memory dynamics," Complexity **2017**, Article ID 2713145 (2017). https://doi.org/10.1155/2017/2713145

34. M. S. Petković, S. Ilić, and J. Džunić, "Derivative-free two-point methods with and without memory for solving nonlinear equations," Appl. Math. Comput. **217**, 1887–1895 (2010).

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