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Necessary and Sufficient Conditions for the Convergence of Two- and Three-Point Newton-Type Iterations

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Abstract—Necessary and sufficient conditions under which two- and three-point iterative methods have the order of convergence p $(2 \le p \le 8)$ are formulated for the first time. These conditions can be effectively used to prove the convergence of iterative methods. In particular, the order of convergence of some known optimal methods is verified using the proposed sufficient convergence tests. The optimal set of parameters making it possible to increase the order of convergence is found. It is shown that the parameters of the known iterative methods with the optimal order of convergence have the same asymptotic behavior. The simplicity of choosing the parameters of the proposed methods is an advantage over the other known methods.

Keywords: nonlinear equations, Newton-type iterations, order of convergence, optimal order.

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1. INTRODUCTION

In recent years, a lot of modified iterative methods were developed that improve the local order of convergence of such methods as Newton's, Ostrowski's, and King's. Among them, the most efficient are the 8th order optimal methods that have the index of efficiency $8^{1/4} \approx 1.682$ (e.g., see [1–9] and references therein). Attempts and comparing these methods from the viewpoint of the behavior of their convergence were also made (see [10, 11]).

Many of these techniques use the optimal Ostrowski or King methods in the first phase and arbitrary real parameters and weighting functions that are difficult to determine. The increase of the convergence order is usually achieved due to additional computations of the function and its derivatives, which can affect the efficiency of the method.

Thus, the development of new optimal methods is presently still important even though many high-order optimal methods are available. To analyze the convergence order, Taylor expansions are typically used, which lead to cumbersome equations and tedious calculations. To overcome these difficulties and find the optimal weighting functions and parameters, symbolic computations and computer algebra systems Mathematica and Maple (see [3, 12]) have been recently used.

In this paper, we propose a novel procedure for the optimization and increasing the order of convergence of computational methods. It is shown that the optimal choice of the parameters of the methods makes it possible to increase the convergence order. In Sections 2 and 3, two-point and three-point iterative methods are considered, and necessary and sufficient conditions for these methods to have the convergence order p for $2 \le p \le 4$ and $5 \le p \le 8$, respectively, are obtained. Numerical experiments supporting the theoretical conclusions on the convergence order and a comparison with other methods are presented in Section 4.

2. TWO-POINT ITERATIVE METHODS

Let x^* be a simple root of the real function $f: I \subset \mathbb{R} \to \mathbb{R}$ defined on an open interval I. Consider the two-point iterative methods

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \tau_n \frac{f(x_n)}{f'(x_n)}, \tag{2.1}$$

where τ_n is the iteration parameter. Note that, in most cases, τ_n in (2.1) takes values ranging from zero to one (see [13]). It was proved in [9] that the continuous analog of the Newton method converges for $\tau_n \in (0, 2)$.

Definition 1. Let $f(x)$ be a real function with the simple root $x^* \in I$, and let $\{x_n\}$ be a sequence of real numbers converging to x^*. The order of convergence p is determined by the condition

$$|x_{n+1} - x^*| \le M |x_n - x^*|^p, \quad p \in \mathbb{R}^+, \quad 0 < M < \infty, \tag{2.2}$$

or by the equivalent condition

$$|f(x_{n+1})| \le C |f(x_n)|^p, \quad 0 < C < \infty. \tag{2.3}$$

Below, we will use (2.3) to analyze the convergence of specific iterative methods.

More precisely, we find the values of the parameter τ_n for which iterations (2.1) give the greatest local order of convergence. To this end, we rewrite (2.1) in the form

$$x_{n+1} = y_n + (\tau_n - 1)(y_n - x_n). \tag{2.4}$$

Using Taylor's expansion of $f(x_{n+1})$ about the point y_n, we obtain

$$f(x_{n+1}) = f(y_n) + f'(y_n)(\tau_n - 1^2)(y_n - x_n) + O(\tau_{n-1}^2(y_n - x_n)^2). \tag{2.5}$$

Let us approximate $f'(y_n)$ using the values of the function calculated before. This can be done using the method of undetermined coefficients so that the condition

$$f'(y_n) = a_n f(x_n) + b_n f(y_n) + c_n f'(x_n) + O(f^2(x_n)) \tag{2.6}$$

is satisfied. Using Taylor's expansion of the functions $f(x_n)$ and $f'(y_n)$ about the point y_n, we obtain the system of linear equations

$$\begin{gathered} a_n + b_n = 0, \\ a_n(x_n - y_n) + c_n = 1, \\ a_n(x_n - y_n)^2 + 2c_n(x_n - y_n) = 0, \end{gathered} \tag{2.7}$$

which has a unique solution

$$a_n = -\frac{2}{y_n - x_n}, \quad b_n = \frac{2}{y_n - x_n}, \quad c_n = -1. \tag{2.8}$$

Substitute (2.8) into (2.6) to obtain

$$f'(y_n) = A_n + O(f^2(x_n)), \tag{2.9}$$

where

$$A_n = 2\frac{f(y_n) - f(x_n)}{y_n - x_n} - f'(x_n). \tag{2.10}$$

Writing $f'(y_n)$ in (2.5) in terms of (2.9), we obtain the formula

$$f(x_{n+1}) = f(x_n)(\theta_n + (\tau_n - 1)(2\theta_n - 1)) + O((\tau_n - 1)^2(y_n - x_n)^2), \quad \theta_n = \frac{f(y_n)}{f(x_n)}. \tag{2.11}$$

We choose τ_n such that the first term on the right-hand side of Eq. (2.11) vanishes, i.e.,

$$\tau_n = \frac{1 - \theta_n}{1 - 2\theta_n}. \tag{2.12}$$

Table 1. Some iterative methods with the order of convergence 4

Methods	τ_n
[14]	$1+\dfrac{\theta_n(1+b\theta_n)}{1+(b-2)\theta_n}$
[15]	$\dfrac{1+\theta_n^2}{1-\theta_n}+\beta\theta_n\dfrac{f^2(x_n)}{f^2(x_n)+(f'(x_n))^2}$
[3]	$1+\theta_n+2\theta_n^2+\theta_n^3$
[16]	$\dfrac{1}{1-\theta_n}+\theta_n^2$

It is seen from (2.1) that

$$f(y_n)=O(f^2(x_n)),\quad y_n-x_n=O(f(x_n)). \tag{2.13}$$

Hence, we have $\theta_n = O(f(x_n))$ and $\tau_n - 1 = O(f(x_n))$. Then, (2.11) implies that

$$|f(x_{n+1})|\le M|f(x_n)|^4$$

under condition (2.12). Therefore, if τ_n is determined by formula (2.12), method (2.1) has the maximal fourth order of convergence.

Using the well-known expansion

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots \tag{2.14}$$

we write (2.12) as

$$\tau_n=1+\theta_n+2\theta_n^2+O(\theta_n^3). \tag{2.15}$$

We formulate a stronger result in the following theorem.

Theorem 1. *Let the function $f(x)$ be sufficiently smooth and have a simple root $x^*\in I$. Furthermore, let the initial approximation x_0 be sufficiently close to x^*. Then, the convergence order of the iterative method* (2.1) *is four if and only if the parameter τ_n satisfies condition* (2.15).

Proof. Assume that τ_n in (2.1) satisfies condition (2.15). Then, according to conditions (2.15) and (2.13), the right-hand side of (2.11) has the order $O(f^4(x_n))$, i.e.,

$$f(x_{n+1})=O(f^4(x_n)).$$

Conversely, let the iterative method (2.1) have the fourth order of convergence. Then, (2.11) implies that

$$\theta_n+(\tau_n-1)(2\theta_n-1)=O(f^3(x_n)).$$

This immediately implies that

$$\tau_n=1+\frac{\theta_n}{1-2\theta_n}+O(f^3(x_n))=1+\theta_n+2\theta_n^2+O(\theta_n^3).$$

Method (2.1) uses the values $f(x_n)$, $f(y_n)$, and $f'(x_n)$ and has the optimal fourth order of convergence. Its index of efficiency is $4^{1/3}\approx 1.587$, which is in agreement with the Kung–Traub conjecture. The known fourth-order methods can be written in form (2.1). The most popular methods are summarized in Table 1. It is easy to verify that all τ_n in Table 1 satisfy condition (2.15). The convergence of every existing method can be proved using the sufficient convergence test (2.15). It is seen from Table 1 that method (2.1), (2.15) can represent, in a certain sense, the class of two-point fourth-order methods.

Theorem 2. *Let all the conditions of Theorem* 1 *be fulfilled. Then, the iterative method* (2.1) *has the convergence order three if and only if the parameter τ_n satisfies the condition*

$$\tau_n=1+\theta_n+O(\theta_n^2). \tag{2.16}$$

Proof. The proof immediately follows from (2.11) and (2.16).

It is clear that the iterative method (2.1) has the convergence order two if $\tau_n \equiv 1 + O(\theta_n)$.

For comparison purposes, consider the iterative method [21]

$$x_{n+1} = \varphi(x_n), \tag{2.17}$$

where $\varphi(x) = x - a_1\omega_1(x) - a_2\omega_2(x) - a_3\omega_3(x)$,

$$\omega_1(x) = \frac{f(x)}{f'(x)}, \qquad \omega_2(x) = \frac{f(x + \beta\omega_1(x))}{f'(x)}, \quad \omega_2(x) = \frac{f(x + \gamma\omega_1(x) + \delta\omega_2(x))}{f'(x)}.$$

In particular, in the case $a_1 = a_2 = 1$, $\beta = -1$, $a_3 = 0$, we can rewrite (2.17) in the form (2.1) with $\tau_n = 1 + \theta_n$. Therefore, according to Theorem 2, this method converges with the order $p = 3$, which is in agreement with the results obtained in [21] (see Table 9.1 in that paper).

3. THREE-POINT ITERATIVE NEWTON-TYPE METHODS

Consider the three-point iterative methods

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad z_n = x_n - \tau_n \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = y_n + t_n(z_n - y_n), \tag{3.1}$$

where τ_n is determined by formula (2.12) an t is the iteration parameter to be determined. As in the preceding section, we rewrite the third expression in (3.1) in the form

$$x_{n+1} = z_n + (t_n - 1)(z_n - y_n). \tag{3.2}$$

Let us use Taylor's expansion of $f \in \mathbb{C}^2(I)$ about the point z_n:

$$f(x_{n+1}) = f(z_n) + f'(z_n)(t_n - 1)(z_n - y_n) + O((t_n - 1)^2(z_n - y_n)^2). \tag{3.3}$$

We approximate $f'(z_n)$ using the values of the functions calculated at the preceding step (see [17]):

$$f'(z_n) = a_n f(x_n) + b_n f(y_n) + c_n f(z_n) + d_n f'(x_n) + O(f^4(x_n)), \tag{3.4}$$

where

$$\begin{gathered} a_n = \frac{\gamma_n(2\gamma_n - 3\omega_n)}{\omega_n(\gamma_n - \omega_n)^2}, \quad b_n = \frac{\omega_n^2}{\gamma_n(\gamma_n - \omega_n)^2}, \\ c_n = -\frac{2\gamma_n + \omega_n}{\gamma_n\omega_n}, \quad d_n = \frac{\gamma_n}{\omega_n - \gamma_n}, \quad \omega_n = x_n - z_n, \quad \gamma_n = y_n - z_n. \end{gathered} \tag{3.5}$$

It is clear from (3.1) that

$$\omega_n = \tau_n \frac{f(x_n)}{f'(x_n)}, \quad \gamma_n = \frac{f(x_n)}{f'(x_n)}(\tau_n - 1), \quad \gamma_n - \omega_n = -\frac{f(x_n)}{f'(x_n)}. \tag{3.6}$$

Substitute (3.6) into (3.5) to obtain

$$a_n = -\frac{(\tau_n + 2)(\tau_n - 1)}{\tau_n}\frac{f'(x_n)}{f(x_n)}, \quad b_n = \frac{\tau_n^2}{\tau_n - 1}\frac{f'(x_n)}{f(x_n)}, \quad c_n = \frac{2 - 3\tau_n}{\tau_n(\tau_n - 1)}\frac{f'(x_n)}{f(x_n)}, \quad d_n = \tau_n - 1. \tag{3.7}$$

Substitute (3.7) into (3.4) and make some transformations to obtain

$$f'(z_n) = A_n + O(f^4(x_n)), \tag{3.8}$$

where

$$A_n = f'(x_n)\left(-\frac{2(\tau_n - 1)}{\tau_n} + \theta_n \frac{\tau_n^2}{\tau_n - 1} + \theta_n \frac{2 - 3\tau_n}{\tau_n(\tau_n - 1)}\frac{f(z_n)}{f(y_n)}\right) \neq 0. \tag{3.9}$$

Replace $f'(z_n)$ in (3.3) by A_n to obtain

$$f(x_{n+1}) = (f(z_n) + A_n(t_n - 1)(z_n - y_n)) + O(f^6(x_n)(t - 1)) + O(f^4(x_n)(t - 1)^4). \tag{3.10}$$

In the derivation of (3.10), we used the following estimates:

$$\tau_n - 1 = O(f(x_n)), \quad z_n - y_n = O(f^2(x_n)). \tag{3.11}$$

Now, choose the parameter t_n such that the first term on the right-hand side of (3.10) vanishes:

$$t_n - 1 = -\frac{f(z_n)}{A_n(z_n - y_n)}. \tag{3.12}$$

Taking into account (3.9), (3.11), and Theorem 1, we obtain from (3.12)

$$t_n - 1 = O(f^2(x_n)). \tag{3.13}$$

Therefore, (3.10) implies

$$f(x_{n+1}) = O(f^8(x_n)) \tag{3.14}$$

under the conditions (2.12) and (3.12).

Typically, the three-point iterative methods of optimal order have the form

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad z_n = y_n - \tilde{\tau}_n\frac{f(y_n)}{f'(x_n)}, \quad x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\alpha_n. \tag{3.15}$$

In particular, the iterative methods (3.1) can be rewritten in the form (3.15) with the parameters

$$\tilde{\tau}_n = \frac{\tau_n - 1}{\theta_n}, \quad \alpha_n = \frac{f'(x_n)}{A_n} = \frac{\tau_n(\tau_n - 1)}{\theta_n\tau_n^3 - 2(\tau_n - 1)^2 + \theta_n(2 - 3\tau_n)\dfrac{f(z_n)}{f(y_n)}}. \tag{3.16}$$

To analyze the order of convergence of iterations (3.15), we use the expansion

$$\tilde{\tau}_n = 1 + 2\theta_n + \beta\theta_n^2 + \gamma\theta_n^3 + \ldots; \tag{3.17}$$

then, we have

$$\tau_n = 1 + \tilde{\tau}_n\theta_n = 1 + \theta_n + 2\theta_n^2 + \beta\theta_n^3 + \gamma\theta_n^4 + \ldots. \tag{3.18}$$

Substitute (3.18) into (3.16) and use expansion (2.14) to obtain

$$\alpha_n = 1 + 2\theta_n + (\beta + 1)\theta_n^2 + (2\beta + \gamma - 4)\theta_n^3 + (1 + 4\theta_n)\frac{f(z_n)}{f(y_n)} + O(\theta_n^4). \tag{3.19}$$

Theorem 3. *Let the conditions of Theorem* 1 *be fulfilled. Then, the convergence order of the iterative method* (3.15) *is eight if and only if the parameters* $\tilde{\tau}_n$ *and* α_n *satisfy conditions* (3.17) *and* (3.19), *respectively.*

Proof. We will use Taylor's expansion of $f(x_{n+1})$ about the point z_n

$$f(x_{n+1}) = f(z_n) - f'(z_n)\frac{f(z_n)}{f'(x_n)}\alpha_n + O(f^2(z_n)) = \left(1 - \frac{A_n}{f'(x_n)}\alpha_n\right)f(z_n) + O(f^2(z_n)), \tag{3.20}$$

taking account of (3.8) and (3.10). Let $\tilde{\tau}_n$ and α_n be determined by formulas (3.17) and (3.19). Then, due to (3.18), condition (2.15) is fulfilled. Therefore, we have by Theorem 1

$$f(z_n) = O(f^4(x_n)).$$

Now, (3.20) with regard to (3.16) implies

$$f(x_{n+1}) = O(f^8(x_n)).$$

Conversely, assume that the order of convergence of iterations (3.15) is 8. Then, (3.20) implies

$$f(z_n) = O(f^4(x_n)) \tag{3.21}$$

and

$$1 - \frac{A_n}{f'(x_n)}\alpha_n = O(f^4(x_n)). \tag{3.22}$$

Table 2. The parameters $\tilde{\tau}_n$ and α_n

Methods	$\tilde{\tau}_n$	α_n
[5]	$\frac{1+b\theta_n}{1+(b-2)\theta_n}$	$\varphi(\theta_n)+\frac{f(z_n)}{f(y_n)-af(z_n)}+4\frac{f(z_n)}{f(x_n)}$, $\varphi(0)=1$, $\varphi'(0)=2$, $\varphi''(0)=10-2b$, $\varphi'''(0)=12b^2-72b+12$
[17]	$\frac{1+b\theta_n}{1+(b-2)\theta_n}$	$\frac{\tilde{\tau}_n(1+\theta_n\tilde{\tau}_n)}{(1+\theta_n\tilde{\tau}_n)^3-2\theta_n\tilde{\tau}_n^2-(1+3\theta_n\tilde{\tau}_n)\frac{f(z_n)}{f(y_n)}}$
[1]	$b=-\frac{1}{2}$	$H(\mu_n)\frac{f'(x_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}$, $\mu_n=\frac{f(z_n)}{f(x_n)}$, $H(0)=1$, $H'(0)=2$, $\lvert H''(0)\rvert<\infty$
[2]	a. $1+\frac{4\theta_n}{2-5\theta_n}$ b. $1+2\theta_n+5\theta_n^2+\theta_n^3$ c. $\frac{1}{1-2\theta_n-\theta_n^2+\theta_n^3}$	$\frac{f(x_n)+\beta f(z_n)}{f(x_n)+(\beta-2)f(z_n)}\frac{f'(x_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}$
[6]	$\frac{1}{1-2\theta_n}$	a. $\frac{1}{2}+\frac{5+8\theta_n+2\theta_n^2}{5-12\theta_n}\left(\frac{1}{2}+\frac{f(z_n)}{f(y_n)}\right)$ b. $\frac{5-2\theta_n+\theta_n^2}{5-12\theta_n}+(1+4\theta_n)\frac{f(z_n)}{f(y_n)}$ c. $1+\frac{4f(z_n)}{f(x_n)+af(z_n)}\left(\frac{1}{1-2\theta_n-\theta_n^2}+\frac{f(z_n)}{f(y_n)}\right)$
[4]	$\frac{1}{1-2\theta_n}$	$\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}+\frac{f(z_n)}{2(f(y_n)-2f(z_n))}\right)^2$ $+\frac{\alpha_1(u_n-z_n)+\alpha_2(y_n-x_n)+\alpha_3(z_n-x_n)}{\beta_1(u_n-z_n)+\beta_2(y_n-x_n)+\beta_3(z_n-x_n)}$, $\alpha_2=\alpha_3$, $\alpha_1=3(\beta_2+\beta_3)\neq 0$
[8]	$\frac{1}{1-2\theta_n}$	$\frac{f'(x_n)f[x_n,y_n]}{f[x_n,z_n]f[y_n,z_n]}w(\mu_n)$, $w(0)=1$, $w'(0)=1$, $\mu_n=\frac{f(z_n)}{f(x_n)}$

Formula (3.21) and Theorem 1 imply that τ_n satisfies condition (2.15) or (3.18). Formula (3.22) implies

$$\alpha_n=\frac{f'(x_n)}{A_n}\left(1+O(f^4(x_n))\right). \tag{3.23}$$

Substitute (3.18) into (3.23) to obtain (3.19).

Remark 1. Methods (3.15) use four values $f(x_n)$, $f(y_n)$, $f(z_n)$, and $f'(x_n)$ at each iteration step and have the optimal order. Their index of efficiency is $8^{1/4}\approx 1.682$.

The known three-point iterative methods can be written in the form (3.15). This allows us to compare them in terms of their computational complexity. It is easy to verify that the parameters $\tilde{\tau}_n$ and α_n, even though they are determined by different formulas, have the same asymptotic behavior (3.17) and (3.19). Thus, the iterative methods (3.15) with the parameters specified by formulas (3.17) and (3.19) completely describe the class of three-point methods of order eight (see Table 2). Thus, the convergence of every existing optimal method can be proved using the sufficient convergence tests (3.17) and (3.19) without using Taylor's expansion. This is the key property of the approach proposed in this paper.

The known three-point iterative methods of the eighth order can be written in the form (3.15) with the parameters $\tilde{\tau}_n$ and α_n. It is easy to verify that all the parameters $\tilde{\tau}_n$ and α_n presented in Table 2 satisfy con-

Table 3. The parameters and for $p = 5$, 6, and 7

p	τ_n	τ_n
7	$1 + \theta_n + 2\theta_n^2 + O(\theta_n^3)$	(3.24)
6	$1 + \theta_n + 2\theta_n^2 + O(\theta_n^3)$	(3.25)
	$1 + \theta_n + O(\theta_n^2)$	(3.24)
5	$1 + \theta_n + 2\theta_n^2 + \beta\theta_n^2 + O(\theta_n^4)$	(3.26)
	$1 + \theta_n + O(\theta_n^2)$	(3.25)

Table 4

Methods	$\tilde{\tau}_n$	α_n	Satisfies	p
[18]	$\frac{1}{1-2\theta_n}$	$\left(\frac{1-\theta_n}{1-2\theta_n}\right)^2 + \frac{f(z_n)}{f(y_n) - \alpha f(z_n)}$	(3.24)	7
[19]	$\frac{1}{1-2\theta_n}$	$1 + 2\theta_n + O(\theta_n^2)$	(3.25)	6
[1]	$\frac{1+b\theta_n}{1+(b-2)\theta_n}$, $b \in \mathbb{R}$	$H(\mu_n)\frac{f'(x_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}$, $\mu_n = \frac{f(z_n)}{f(x_n)}$, $H(0) = 1$	(3.24) for $\beta = 4 - 2b$	7
[20]	$\frac{1}{(1-\theta_n)^2}$	$\frac{1}{\left(1 - \theta_n\left(1 + \frac{f(z_n)}{f(y_n)}\right)\right)^2}$	(3.25)	6

ditions (3.17) and (3.19). Therefore, the convergence of every existing method of order eight can be proved using the sufficient convergence criteria (3.17) and (3.19). This gives a new simple method for establishing the convergence order of thee-point iterative methods. Note that conditions (3.17) for $\tilde{\tau}_n$ is usually satisfied.

Theorem 4. *Let the conditions of Theorem* 1 *be fulfilled. Then, the convergence order of the iterative method* (3.15) *is p if and only if the parameters* τ_n *and* α_n *are given by Table* 3.

Proof. The proof immediately follows from (3.20).

In Table 3, the following formulas were used:

$$\alpha_n = 1 + 2\theta_n + (\beta + 1)\theta_n^2 + \frac{f(z_n)}{f(y_n)} + O(\theta_n^3), \tag{3.24}$$

$$\alpha_n = 1 + 2\theta_n + O(\theta_n^2), \tag{3.25}$$

$$\alpha_n = 1 + O(\theta_n). \tag{3.26}$$

Table 4 shows the list of the parameters $\tilde{\tau}_n$ and α_n for some known methods. The simplest choice (3.19) is called optimal. It is clear that the iterative methods (3.15) with the optimal order are the best ones among the methods with different parameters in terms of the computational cost (see Tables 2 and 3). In Tables 2 and 5, the first and the second divided differences were used

$$f[x, y] = \frac{f(y) - f(x)}{y - x}, \quad f[x, y, z] = \frac{f(y, z) - f(x, y)}{z - x}.$$

Table 5. Two-point iterative methods (2.1) in the case $\varepsilon = 10^{-150}$. Here x_0 is the initial approximation, and l in parentheses denotes 10^l

τ_n	$\lvert x^* - x_n\rvert$	$\lvert f(x_n)\rvert$	n	COC	$\lvert x^* - x_n\rvert$	$\lvert f(x_n)\rvert$	n	COC
	$f_1(x), x_0 = 3.1$				$f_2(x), x_0 = -1.3$			
$1 + \theta_n + 2\theta_n^2$	1.509 (−190)	1.961 (−189)	5	4.000	6.879 (−178)	1.397 (−176)	4	4.000
$1 + \theta_n$	4.731 (−155)	6.150 (−154)	6	3.000	1.049 (−179)	2.132 (−178)	5	3.000
1	1.258 (−297)	1.016 (−295)	10	2.000	1.258 (−222)	2.555 (−221)	8	2.000
	$f_3(x), x_0 = 2.0$				$f_4(x), x_0 = 1.0$			
$1 + \theta_n + 2\theta_n^2$	1.696 (−428)	4.689 (−428)	6	4.000	2.710 (−228)	1.004 (−226)	6	4.000
$1 + \theta_n$	2.000 (−188)	5.527 (−188)	7	3.000	3.622 (−401)	1.342 (−399)	8	3.000
1	4.719 (−219)	1.304 (−218)	9	2.000	3.214 (−193)	1.190 (−191)	9	2.000
	$f_5(x), x_0 = 1.9$				$f_6(x), x_0 = 2.4$			
$1 + \theta_n + 2\theta_n^2$	3.113 (−257)	1.868 (−256)	5	4.000	6.026 (−201)	8.392 (−200)	4	4.000
$1 + \theta_n$	8.313 (−243)	4.988 (−242)	6	3.000	1.362 (−205)	1.896 (−204)	5	3.000
1	2.706 (−285)	1.623 (−284)	9	2.000	3.437 (−250)	4.787 (−249)	8	2.000

4. NUMERICAL EXPERIMENTS

Consider the examples discussed in [7, 17]:

$$f_1(x) = \exp(x^2 + 7x - 30) - 1, \quad x^* = 3,$$

$$f_2(x) = x\exp(x^2) - \sin^2(x) + 3\cos(x) + 5, \quad x^* \approx -1.20764782\ldots,$$

$$f_3(x) = 10x\exp(-x^2) - 1, \quad x^* \approx 1.67963061\ldots,$$

$$f_4(x) = x^5 + x^4 + 4x^2 - 15, \quad x^* \approx 1.34742809\ldots,$$

$$f_5(x) = (x-1)^6 - 1, \quad x^* = 2,$$

$$f_6(x) = x^3 - 10, \quad x^* \approx 2.15443469\ldots.$$

All the computations were performed using Maple 18. To check the convergence, we calculate the computational order of convergence (COC) by the formulas

$$d_{x_n} = \frac{\ln(|x_n - x^*|/|x_{n-1} - x^*|)}{\ln(|x_{n-1} - x^*|/|x_{n-2} - x^*|)},$$

where x_n, x_{n-1}, x_{n-2} are three successive approximations.

The iteration process is stopped when

$$|f(x_n)| \le \varepsilon.$$

The errors of computing $f_i(x) = 0$ ($i = 1, 2, \ldots, 6$) and the approximation order are presented in Tables 5 and 6. It is seen that the actual order of convergence perfectly coincides with the theoretical order of convergence. Table 7 shows the comparison results of methods (2.1) and (3.15) with the methods proposed in [1, 16, 19]. It is seen from Table 7 that the order of convergence and the required number of iterations are the same for these methods. The methods proposed in [1, 19] are more complicated than (3.15) because the implementation of the methods [1, 19] requires a greater number of arithmetic operations.

Table 6. Three-point iterative methods (3.15). Here $\beta = 2(2-b)$, $\gamma = 2(2-b)^2$, $\varepsilon = 10^{-150}$, x_0 is the initial approximation, and l in parentheses denotes 10^l

$\tilde{\tau}_n = 1 + 2\theta_n + \beta\theta_n^2 + \gamma\theta_n^3$		$\lvert x^* - x_n\rvert$	$\lvert f(x_n)\rvert$	n	COC	$\lvert x^* - x_n\rvert$	$\lvert f(x_n)\rvert$	n	COC
α_n	b	$f_1(x)$, $x_0 = 3.1$				$f_2(x)$, $x_0 = -1.3$			
(3.19)	−1	2.30 (−1096)	3.00 (−1095)	4	8.000	9.59 (−377)	1.94 (−375)	3	8.000
	0	4.17 (−878)	5.42 (−877)	4	8.000	1.24 (−418)	2.52 (−417)	3	8.000
	1	2.74 (−690)	3.57 (−689)	4	8.000	3.97 (−344)	8.07 (−343)	3	7.999
(3.24)	−1	1.31 (−617)	1.70 (−618)	4	7.000	2.85 (−229)	5.78 (−228)	3	7.000
	0	8.73 (−515)	1.13 (−513)	4	7.000	8.36 (−278)	1.69 (−276)	3	6.999
	1	1.08 (−412)	1.40 (−411)	4	7.000	1.52 (−231)	3.08 (−230)	3	6.999
(3.25)	−1	1.41 (−368)	1.84 (−367)	4	6.000	1.11 (−152)	2.25 (−151)	3	6.000
	0	3.64 (−301)	4.74 (−300)	4	6.000	1.55 (−185)	3.15 (−184)	3	6.000
	1	2.99 (−232)	3.88 (−231)	4	6.000	1.11 (−152)	2.25 (−151)	3	6.000
(3.26)	−1	6.33 (−208)	8.24 (−207)	4	5.000	6.76 (−469)	1.37 (−467)	4	5.000
	0	1.18 (−171)	1.54 (−170)	4	5.000	7.43 (−591)	4.50 (−589)	4	5.000
	1	5.19 (−626)	6.74 (−625)	5	5.000	6.76 (−469)	1.37 (−467)	4	5.000
		$f_3(x)$, $x_0 = 1.0$				$f_4(x)$, $x_0 = 2.0$			
(3.19)	−1	3.84 (−164)	1.06 (−163)	3	8.000	1.18 (−1082)	4.40 (−1081)	4	8.000
	0	9.85 (−165)	2.72 (−164)	3	8.000	1.03 (−889)	3.82 (−888)	4	8.000
	1	7.22 (−159)	1.99 (−158)	3	8.000	2.32 (−718)	8.62 (−717)	4	8.000
(3.24)	−1	2.17 (−964)	6.00 (−964)	4	7.000	2.79 (−615)	1.03 (−613)	4	7.000
	0	3.90 (−971)	1.07 (−970)	4	7.000	1.90 (−523)	7.07 (−522)	4	7.000
	1	2.95 (−941)	8.17 (−941)	4	7.000	9.03 (−431)	3.34 (−429)	4	7.000
(3.25)	−1	3.77 (−457)	1.04 (−456)	4	6.000	1.22 (−364)	4.52 (−363)	4	6.000
	0	1.19 (−459)	3.29 (−459)	4	6.000	1.47 (−304)	5.47 (−303)	4	6.000
	1	4.27 (−437)	1.18 (−436)	4	6.000	4.64 (−242)	1.71 (−240)	4	6.000
(3.26)	−1	1.07 (−287)	2.97 (−287)	4	5.000	6.03 (−205)	2.23 (−203)	4	5.000
	0	1.52 (−289)	4.22 (−289)	4	5.000	3.51 (−172)	1.30 (−170)	4	5.000
	1	1.98 (−273)	5.47 (−273)	4	5.000	2.72 (−653)	1.00 (−651)	5	5.000
		$f_5(x)$, $x_0 = 2.1$				$f_6(x)$, $x_0 = 2.4$			
(3.19)	−1	4.68 (−293)	2.81 (−292)	3	8.000	2.21 (−426)	3.08 (−425)	3	8.000
	0	1.01 (−278)	6.08 (−278)	3	7.999	7.10 (−427)	9.88 (−426)	3	8.000
	1	5.66 (−238)	3.40 (−237)	3	7.999	4.05 (−382)	5.65 (−381)	3	7.999
(3.24)	−1	8.10 (−185)	4.86 (−184)	3	7.000	4.91 (−274)	6.84 (−273)	3	7.000
	0	5.35 (−186)	3.21 (−185)	3	6.999	4.91 (−285)	6.84 (−284)	3	6.999
	1	5.20 (−162)	3.12 (−161)	3	6.999	5.05 (−259)	7.03 (−258)	3	6.999
(3.25)	−1	3.67 (−742)	2.20 (−741)	4	6.000	6.23 (−183)	8.68 (−182)	3	6.000
	0	3.23 (−737)	1.94 (−736)	4	6.000	1.69 (−187)	2.36 (−186)	3	5.999
	1	2.27 (−624)	1.36 (−623)	4	5.999	1.05 (−166)	1.47 (−165)	3	5.999
(3.26)	−1	6.32 (−396)	3.79 (−395)	4	5.000	6.26 (−569)	8.72 (−568)	4	5.000
	0	1.09 (−395)	6.55 (−395)	4	5.000	1.25 (−586)	1.75 (−585)	4	5.000
	1	1.55 (−325)	9.34 (−325)	4	5.000	3.02 (−510)	4.21 (−509)	4	5.000

Table 7. Comparison of iterative methods: $\varepsilon = 10^{-150}$, x_0 is the initial approximation, l in parentheses denotes 10^l

Methods	$f_1(x),\ x_0 = 3.1$				$f_2(x),\ x_0 = -1.3$			
	$\lvert x^* - x_n\rvert$	$\lvert f_1(x_n)\rvert$	n	COC	$\lvert x^* - x_n\rvert$	$\lvert f_1(x_n)\rvert$	n	COC
(2.1), $\tau_n = 1 + \theta_n + 2\theta_n^2$	1.509(−190)	1.961(−189)	5	4.000	6.879(−178)	1.397(−176)	4	4.000
[16]	5.231(−210)	6.801(−209)	5	4.000	7.744(−187)	1.573(−185)	4	4.000
(3.15), $\tau_n = 1 + 2\theta_n + 6\theta_n^2 + 18\theta_n^3$ $\alpha_n = 1 + 2\theta_n$	1.419(−368)	1.844(−367)	4	6.000	1.110(−152)	2.254(−151)	3	6.000
[19], $H(t) = \dfrac{\gamma + (\beta + 2\gamma)t}{\gamma + \beta t}$, $\beta = \gamma = 1$	3.020(−342)	3.926(−341)	4	6.000	1.468(−192)	2.981(−191)	3	5.999
(3.15), $\tau_n = 1 + 2\theta_n + 6\theta_n^2 + 18\theta_n^3$ $\alpha_n = 1 + 2\theta_n + 7\theta_n^2 + 26\theta_n^3 + (1 + 4\theta_n)\dfrac{f(z_n)}{f(x_n)}$	2.308(−1096)	3.001(−1095)	4	8.000	9.593(−377)	1.948(−375)	3	8.000
[1], (36), $\alpha = 1$	4.055(−688)	5.272(−687)	4	8.000	2.326(−399)	4.724(−398)	3	8.000

5. CONCLUSIONS

Novel necessary and sufficient conditions for two-and three-point iterative methods to have the convergence order p $(2 \le p \le 8)$ are obtained.

Simple (including the optimal) sets of parameters of the iterative procedures are proposed that make it possible to increase the order of convergence. These results are new in the theory of such iterative processes. Probably, other iterative methods of types (2.1) and (3.15) can be proposed. But all of them will differ only in the choice of the parameters that have the same asymptotic behavior. In this sense, Theorems 1 and 3 answer the question of how the iterative methods (2.1) and (3.15) with the optimal convergence order can be constructed. Only the question of their comparison from the viewpoint of the behavior of convergence remains open.

An advantage of the proposed approach is that there is no need to obtain cumbersome equations for the error: the order of convergence of methods (2.1) and (3.15) can be established using the sufficient tests obtained in this paper.

The simplicity of choosing the parameters is another advantage over the other known methods (see Tables 1 and 2).

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