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Two-sided approximation for some Newton's type methods

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ABSTRACT

We suggest and analyze a combination of a damped Newton's method and a simplified version of Newton's one. We show that the proposed iterations give two-sided approximations of the solution which can be efficiently used as posterior estimations. Some numerical examples illustrate the efficiency and performance of the method proposed.

1. Introduction

In the last decade, new iterative methods containing parameters for a numerical solving of nonlinear equations have been developed by many authors. The role of these parameters play, for example, a damped parameter in Newton type methods [1–6], interpolation nodes in inverse polynomial interpolation methods [7,8]. They can be controlled not only by the convergence order, but also by the convergence behavior. One of the advantages of such methods is that they give two-sided approximations of the solutions which allow one to control the error at each iteration step [8,3,6]. In this paper we will consider a combination of the damped Newton's method and the simplified Newton method.

The paper is organized as follows. In Section 2 we formulate new iteration schemes for solving nonlinear equations. In Section 3 we show that the proposed iterations give two-sided approximation of the solution. In Section 4 we prove that the convergence order of these iterations is at least 2. Depending on the suitable choices of parameters, the convergence order may be increased from 2 to 4. Some numerical examples illustrating the theoretical results are given in Section 5.

2. Statement of the problem

Let $a, b \in \mathbb{R},\ a < b,\ f : [a, b] \to \mathbb{R}$ and consider the following nonlinear equation

$$f(x) = 0. \tag{1}$$

Assume that $f(x) \in C^3[a, b],\ f'(x) \neq 0,\ x \in [a, b]$ and Eq. (1) has a unique root $x^* \in [a, b]$. For a numerical solution of Eq. (1) we propose the following iterations

$$x_{2n+1} = x_{2n} - \tau_n \frac{f(x_{2n})}{f'(x_{2n})}, \quad n = 0, 1, \ldots, \tag{2a}$$

$$x_{2n+2} = x_{2n+1} - \omega_n f(x_{2n+1}). \tag{2b}$$

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Here $\tau_n > 0$ and ω_n are the iteration parameters to be determined properly. It should be mentioned that the first iteration (2a) is a continuous analogy of Newton's method (or damped Newton's method), while the second one (2b) is a simple iteration. In [6] it is shown that the iterations (2a) and (2b) with

$$\omega_n = \frac{1}{f'(x_{2n+1})} \tag{3}$$

have a two-sided approximation behavior, and it proved the convergence rate of these iterations is 4 when $\tau_n \to 1$ as $n \to \infty$.

On the other hand, the iterations (2a) and (2b) can be considered as simple iterations

$$x_{2n+1} = p(x_{2n}), \quad x_{2n+2} = q(x_{2n+1}), \quad n = 0, 1, \ldots, \tag{4}$$

for two equations

$$x - p(x) = 0, \quad x - q(x) = 0, \tag{5}$$

which are equivalent to the above Eq. (1) and with functions

$$p(x) = x - \tau \frac{f(x)}{f'(x)}, \quad q(x) = x - \omega f(x). \tag{6}$$

3. The convergence of the proposed iterations

Suppose that [6]

$$\left| \frac{f''(x)}{(f'(x))^2} f(x) \right| \leqslant M_2 \left| \frac{f(x)}{(f'(x))^2} \right| \leqslant a(x) < \frac{4}{9}, \quad x \in [a, b], \tag{7}$$

where $M_2 = \max_{x\in[a,b]} |f''(x)|$. Then it is easy to show that the function $p(x)$ satisfies

$$0 < p'(x_{2n}) < 1, \quad n = 0, 1, \ldots, \tag{8}$$

under condition

$$\tau_n \in \left(0, \frac{1}{1 - a_{2n}}\right), \quad a_{2n} = M_2 \left| \frac{f(x_{2n})}{(f'(x_{2n}))^2} \right|. \tag{9}$$

A sufficient condition for $q(x)$ to be decreasing is

$$\omega_n f'(x_{2n+1}) > 1, \quad n = 0, 1, \ldots \tag{10}$$

It should be noted that the conditions (8) and (10) were used first in [7,8] for bilateral approximations of Aitken–Steffensen–Hermite type methods.

Using Taylor expansion of $f(x_{2n+2})$ at point x_{2n+1}, and (2b), we obtain

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} = 1 - \omega_n f'(x_{2n+1}) + \frac{f''(\xi_{2n})}{2} f(x_{2n+1}) \omega_n^2, \tag{11}$$

where $\xi_{2n} = \theta x_{2n+2} + (1 - \theta) x_{2n+1}, \ \theta \in (0, 1)$.

Lemma 1. *Suppose that*

$$f''(\xi_{2n}) f(x_{2n+1}) < 0, \quad n = 0, 1 \tag{12}$$

and the inequality (10) *holds. Then*

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} < 0, \quad n = 0, 1, \ldots \tag{13}$$

Proof. If to take (12) and (10) into account, then from formula (11) we get

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} < 1 - \omega_n f'(x_{2n+1}) < 0. \tag{14}$$

The Lemma is proved. □

Analogously, using Taylor expansion of $f(x_{2n+1})$ at point x_{2n}, and (2a) we obtain

$$\frac{f(x_{2n+1})}{f(x_{2n})} = 1 - \tau_n + \frac{f''(\eta_{2n})}{2} \frac{f(x_{2n})}{(f'(x_{2n}))^2} \tau_n^2, \quad n = 0, 1, \ldots, \tag{15}$$

where $\eta_{2n} = \alpha x_{2n+1} + (1-\alpha)x_{2n}, \alpha \in (0,1)$.

Lemma 2. *Suppose that the inequality (7) holds. Then*

$$\frac{f(x_{2n+1})}{f(x_{2n})} < 0, \quad n = 0, 1, \ldots, \tag{16}$$

under condition

$$\tau_n \in I_{2n} = \left[\frac{1-\sqrt{1-2a_{2n}}}{a_{2n}}, \frac{-1+\sqrt{1+4a_{2n}}}{a_{2n}}\right) \subseteq [1,2). \tag{17}$$

Proof. From (15) we obtain

$$\frac{f(x_{2n+1})}{f(x_{2n})} < 1 - \tau_n + \frac{a_{2n}}{2}\tau_n^2 \le 0. \tag{18}$$

From this it follows the condition (16) holds if

$$\tau_n \in \left[\frac{1-\sqrt{1-2a_{2n}}}{a_{2n}}, \frac{1+\sqrt{1-2a_{2n}}}{a_{2n}}\right]. \tag{19}$$

On the other hand, as shown in [5], the iteration parameter τ_n must be taken from the τ-region of convergence of iteration (2a)

$$\tau_n \in \left(0, \frac{-1+\sqrt{1+4a_{2n}}}{a_{2n}}\right) \subseteq (0,\ 2). \tag{20}$$

Hence, the inequality (16) is valid for $\tau_n \in I_{2n}$, and I_{2n} is not an empty interval because of (7). The Lemma is proved. □

We obtained the following results when $f(x)$ is increasing and convex on the interval $x \in [a,b]$.

Theorem 1. *Let $x_0 \in (x^*, b]$, and $f(x)$ satisfies the following conditions:*

(i_1) $f'(x) > 0,\ x \in [a,b]$,
(ii_1) $f''(x) > 0,\ x \in [a,b]$,
(iii_1) *the inequality (7) holds.*

If the parameters τ_n and ω_n are chosen such that

$$\tau_n \in \left(0, \frac{1}{1-a_{2n}}\right) \cap I_{2n}, \quad n = 0, 1, \ldots \tag{21}$$

and

$$\omega_n f'(a) \geqslant 1, \quad n = 0, 1, \ldots, \tag{22}$$

then the following relations hold:

(j_1) $x_1 < x_3 < \cdots < x_{2n+1} < x^* < x_{2n} < \cdots < x_2 < x_0$,
(jj_1) $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n} = x^*$.

Proof. By (i_1) it follows that $x^* \in (a,b)$ is the unique solution of Eq. (1). From (8)–(10) the functions $p(x)$ and $q(x)$ have no extremum on the own domain of definition, and $p(a) > a,\ p(b) < b,\ q(a) > a,\ q(b) < b$ because of $f(a) < 0$ and $f(b) > 0$. Therefore, all the approximations generated by (2a) and (2b) belong to $[a,b]$, i.e. $x_{2n+1}, x_{2n+2} \in (a,b),\ n = 0, 1, \ldots$ By assumption of theorem $f(x_0) > 0$. Then, according to Lemma 2, from (16) it follows that $f(x_1) < 0$. By (ii_1), $f'(x)$ is increasing on the interval $[a,b]$, i.e.

$$0 < f'(a) < f'(x) < f'(b), \quad x \in (a,b). \tag{23}$$

Therefore,

$$\frac{1}{f'(a)} > \frac{1}{f'(x)} > \frac{1}{f'(b)}, \quad x \in (a,b). \tag{24}$$

According to (22), we have

$$\omega_0 \geqslant \frac{1}{f'(a)} > \frac{1}{f'(x)}, \quad x \in (a,b), \tag{25}$$

which holds, for instance, for x_1, i.e. the condition (10) is fulfilled for $n = 0$. By virtue of (ii_1) and $f(x_1) < 0$, the assumption (12) is valid for $n = 0$. Then, according to Lemma 1, from (13) we obtain $f(x_2) > 0$. By induction on n from (13) and (16) one can show that

$$f(x_{2n}) > 0, \quad f(x_{2n+1}) < 0 \tag{26}$$

and also can prove (8) and (10) for all $n = 0, 1, \ldots$

Thus, the sequence $\{x_{2n+1}\}$ generated by (2a) (or (4)) is increasing and the sequence $\{x_{2n+2}\}$ generated by (2b) (or (4)) is decreasing. Consequently we have

$$x_1 < x_3 < \cdots < x_{2n+1} < x^* < x_{2n} < \cdots < x_2 < x_0, \tag{27}$$

i.e. (j_1) is proved. The (jj_1) follows from (j_1) passing to the limit $n \to \infty$. Since $f(x)$ is increasing function on the interval $[a,b]$, then we have

$$f(x_1) < f(x_3) < \cdots < f(x_{2n+1}) < f(x^*) < f(x_{2n}) < \cdots < f(x_2) < f(x_0) \tag{28}$$

and

$$\lim_{n\to\infty} f(x_{2n+1}) = \lim_{n\to\infty} f(x_{2n}) = f(x^*) = 0. \tag{29}$$

It should be pointed out that the intersection of two intervals in (21) is not an empty set because of (7). The Theorem is proved. □

If $f(x)$ is decreasing and concave on the interval $x \in [a,b]$, instead of Eq. (1), we consider the equation

$$-f(x) = 0. \tag{30}$$

The function $-f(x)$ satisfies the conditions of Theorem 1. From here we have:

Corollary 1. *Let $x_0 \in (x^*, b]$, and $f(x)$ satisfies the following conditions:*

(i_2) $f'(x) < 0,\ x \in [a,b]$,
(ii_2) $f''(x) < 0,\ x \in [a,b]$,
(iii_2) *the inequality (7) holds.*

If the parameter τ_n is taken from the interval (21) and parameter ω_n is chosen such that (22), then the relations (j_1) and (jj_1) hold too.

When

$$f''(x)f'(x) < 0, \quad x \in [a,b], \tag{31}$$

we consider instead of iterations (2a) and (2b) the following iterations

$$x_{2n+1} = x_{2n} - \omega_n f(x_{2n}), \quad n = 0, 1, \ldots, \tag{32a}$$

$$x_{2n+2} = x_{2n+1} - \tau_n \frac{f(x_{2n+1})}{f'(x_{2n+1})}. \tag{32b}$$

As above, the iterations (32a) and (32b) can be considered as

$$x_{2n+1} = q(x_{2n}), \quad x_{2n+2} = p(x_{2n+1}), \tag{33}$$

with functions $q(x)$ and $p(x)$ given by (6).

Using Taylor expansion of $f(x_{2n+2})$ at point x_{2n+1}, and (32b), we obtain

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} = 1 - \tau_n + \frac{f''(\xi^*_{2n+2})}{2} \frac{f(x_{2n+1})}{(f'(x_{2n+1}))^2} \tau_n^2, \tag{34}$$

where $\xi^*_{2n+2} = \theta^* x_{2n+2} + (1-\theta^*)x_{2n+1}, \quad \theta^* \in (0,1)$.

Lemma 3. *Suppose that*

$$f''(\xi^*_{2n+2}) f(x_{2n+1}) < 0, \quad n = 0, 1, \ldots \tag{35}$$

Then the inequality

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} < 0, \quad n = 0, 1, \ldots \tag{36}$$

holds for any $\tau_n \geqslant 1$.

Proof. The inequality (36) immediately follows from (34), if we take into account (35) and $\tau_n \geqslant 1$. The Lemma is proved. □

Analogously, using Taylor expansion of $f(x_{2n+1})$ at point x_{2n}, and (32a), we obtain

$$\frac{f(x_{2n+1})}{f(x_{2n})} = 1 - \omega_n f'(x_{2n}) + \frac{f''(\eta^*_{2n})}{2} f(x_{2n})\omega_n^2, \tag{37}$$

where $\eta^*_{2n} = \alpha^* x_{2n+1} + (1-\alpha^*)x_{2n},\ \alpha^* \in (0,1)$.

Lemma 4. *Suppose that the inequality (7) holds and*

$$f''(\eta^*_{2n})f(x_{2n}) > 0, \quad n = 0,1,\ldots \tag{38}$$

Then

$$\frac{f(x_{2n+1})}{f(x_{2n})} < 0, \quad n = 0,1,\ldots, \tag{39}$$

under condition

$$\omega_n f'(x_{2n}) \in I_{2n}, \quad n = 0,1,\ldots \tag{40}$$

Proof. The proof of Lemma 4 is the same as the proof of Lemma 2. □

Now we are ready to prove the following theorem when $f(x)$ is increasing and concave on the interval $x \in [a,b]$.

Theorem 2. *Let* $x_0 \in [a, x^*)$, *and* $f(x)$ *satisfies the following conditions*

(i_3) $f'(x) > 0,\ x \in [a,b]$,
(ii_3) $f''(x) < 0,\ x \in [a,b]$,
(iii_3) *the inequality (7) holds.*

If the parameters τ_n *and* ω_n *are chosen such that*

$$\tau_n \in \left[1, \frac{1}{1-a_{2n+1}}\right), \quad a_{2n+1} = M_2\left|\frac{f(x_{2n+1})}{(f'(x_{2n+1}))^2}\right|, \quad n = 0,1,\ldots \tag{41}$$

and

$$\omega_n f'(x_{2n}) \in I_{2n}, \quad n = 0,1,\ldots, \tag{42}$$

then the following relations hold:

(j_3) $x_0 < x_2 < \cdots < x_{2n} < x^* < x_{2n+1} < \cdots < x_3 < x_1$,
(jj_3) $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = x^*$.

Proof. By (i_3) it follows that $x^* \in (a,b)$ is the unique solution of Eq. (1) and $f(x_0) < 0$. By (ii_3) and $f(x_0) < 0$ the assumption of Lemma 4 is fulfilled for $n = 0$. Then from (39) it follows that $f(x_1) > 0$ under condition (42). The assumption (35) is valid for $n = 1$. Then by Lemma 3 the inequality (36) is valid for $n = 1$, i.e., $f(x_2) < 0$ under condition (41). By induction on n from (36) and (39) one can prove that

$$f(x_{2n}) < 0, \quad f(x_{2n+1}) > 0 \tag{43}$$

and

$$q'(x_{2n}) = 1 - \omega_n f'(x_{2n}) \leq 0, \quad 0 < p'(x_{2n+1}) < 1. \tag{44}$$

Therefore, we have

$$x_0 < x_2 < \cdots < x_{2n} < x^* < x_{2n+1} < \cdots < x_3 < x_1. \tag{45}$$

The (j_3) is proved. The (jj_3) follows from (j_3) passing to the limit $n \to \infty$. Since $f(x)$ is increasing on $[a,b]$, then from (j_3) it follows that

$$f(x_0) < f(x_2) < \cdots < f(x_{2n}) < f(x^*) < f(x_{2n+1}) < \cdots < f(x_3) < f(x_1) \tag{46}$$

and

$$\lim_{n\to\infty} f(x_{2n}) = \lim_{n\to\infty} f(x_{2n+1}) = f(x^*) = 0. \tag{47}$$

The Theorem is proved. □

If $f(x)$ is decreasing and convex on the interval $x \in [a,b]$, the function $-f(x)$ satisfies the conditions of Theorem 2. From here we have:

Corollary 2. *Let $x_0 \in [a, x^*)$, and $f(x)$ satisfies the following conditions*

(i$_4$) $f'(x) < 0,\ x \in [a,b]$,
(ii$_4$) $f''(x) > 0,\ x \in [a,b]$,
(iii$_4$) *the inequality (7) holds.*

If the parameter τ_n is taken from the interval (41) and parameter ω_n is chosen such that (42), then the relations (j$_3$) and (jj$_3$) hold, too.

4. The convergence order of proposed iterations

The convergence order of proposed iterations (2a), (2b) and (32a), (32b) is given in the following results.

Theorem 3. *Assume that $f(x) \in C^4[a,b],\ f'(x) \neq 0,\ x \in [a,b]$, and there exists a unique solution $x^* \in [a,b]$ of Eq. (1). Then the q-convergence (or p-convergence) order of the sequence $\{x_n\}$ generated by iterations (2a) and (2b) (or (32a) and (32b)) is at least 2, when $\tau_n \to 1$ as $n \to \infty$.*

Proof. First we consider the convergence order of iterations (2a) and (2b). Let $e_n = x_n - x^*$. From (2a) and (2b) it follows that

$$e_{2n+1} = e_{2n} - \tau_n \frac{f(x_{2n}) - f(x^*)}{f'(x_{2n})}, \tag{48a}$$

$$e_{2n+2} = e_{2n+1} - \omega_n (f(x_{2n+1}) - f(x^*)). \tag{48b}$$

Using the Taylor expansions for $f(x^*)$ at points x_{2n} and x_{2n+1} in (48a) and (48b), respectively, we obtain

$$e_{2n+1} = e_{2n}\left\{1 - \tau_n + \tau_n \frac{f''(x_{2n})}{2f'(x_{2n})} e_{2n} - \tau_n \frac{f'''(x_{2n})}{6f'(x_{2n})} e_{2n}^2\right\} + O(e_{2n}^4), \tag{49a}$$

$$e_{2n+2} = e_{2n+1}\left\{1 - \omega_n f'(x_{2n+1}) + \omega_n \frac{f''(x_{2n+1})}{2} e_{2n+1} - \omega_n \frac{f'''(x_{2n+1})}{6} e_{2n+1}^2\right\} + O(e_{2n+1}^4). \tag{49b}$$

Substituting e_{2n+1} from (49a) into (49b), we have

$$\begin{aligned} e_{2n+2} = {} & (1-\tau_n)(1-\omega_n f'(x_{2n+1}))e_{2n} + \left\{(1-\tau_n)^2 \omega_n \frac{f''(x_{2n+1})}{2} + \tau_n \frac{f''(x_{2n})}{2f'(x_{2n})}(1-\omega_n f'(x_{2n+1}))\right\} e_{2n}^2 \\ & + \left\{-\tau_n \frac{f'''(x_{2n})}{6f'(x_{2n})}(1-\omega_n f'(x_{2n+1})) + (1-\tau_n)\tau_n\omega_n \frac{f''(x_{2n+1})f''(x_{2n})}{2f'(x_{2n})} - (1-\tau_n)^3 \omega_n \frac{f'''(x_{2n+1})}{6}\right\} e_{2n}^3 \\ & + O(e_{2n}^4) + O(e_{2n+1}^4). \end{aligned} \tag{50}$$

When $\tau_n \to 1$ as $n \to \infty$, from (50) we conclude that

$$\begin{aligned} & e_{2n+2} = O(e_{2n}^2), \quad \text{if } \omega_n = const, \\ & e_{2n+2} = O(e_{2n}^3), \quad \text{if } \omega_n - (f'(x_{2n+1}))^{-1} = O(e_{2n}), \\ & e_{2n+2} = O(e_{2n}^4), \quad \text{if } \omega_n = (f'(x_{2n+1}))^{-1}. \end{aligned} \tag{51}$$

Using similar calculations for iterations (32a) and (32b), we obtain

$$\begin{aligned} e_{2n+2} = {} & (1-\tau_n)(1-\omega_n f'(x_{2n}))e_{2n} + \left\{(1-\tau_n)\omega_n \frac{f''(x_{2n})}{2} + \tau_n \frac{f''(x_{2n+1})}{2f'(x_{2n+1})}(1-\omega_n f'(x_{2n}))^2\right\} e_{2n}^2 \\ & + \left\{-\tau_n \frac{f'''(x_{2n+1})}{6f'(x_{2n+1})}(1-\omega_n f'(x_{2n}))^3 + \tau_n\omega_n \frac{f''(x_{2n+1})f''(x_{2n})}{2f'(x_{2n+1})}(1-\omega_n f'(x_{2n})) - (1-\tau_n)\omega_n \frac{f'''(x_{2n})}{6}\right\} e_{2n}^3 \\ & + O(e_{2n}^4) + O(e_{2n+1}^4). \end{aligned} \tag{52}$$

When $\tau_n \to 1$ as $n \to \infty$, from (52) we have

$$e_{2n+2} = O(e_{2n}^2), \quad \text{if } \omega_n = const,$$
$$e_{2n+2} = O(e_{2n}^4), \quad \text{if } \omega_n - (f'(x_{2n}))^{-1} = O(e_{2n}), \tag{53}$$
$$e_{2n+2} = O(e_{2n}^4), \quad \text{if } \omega_n = (f'(x_{2n}))^{-1}.$$

Thus, the q-convergence (or p-convergence) order of iterations (2a) and (2b) (or (32a) and (32b)) is at least 2, which completes the proof of theorem. □

From Theorems 1–3 and Corollaries 1, 2 it is clear that the best choice of parameters are

$$\tau_n = \frac{1-\sqrt{1-2a_{2n}}}{a_{2n}}\bigg|_{n\to\infty} \to 1, \quad \omega_n = \frac{1}{f'(x_{2n+1})}, \tag{54}$$

for iterations (2a) and (2b) and

$$\tau_n = 1, \quad \omega_n = \frac{1-\sqrt{1-2a_{2n}}}{a_{2n}f'(x_{2n})}\bigg|_{n\to\infty} \to \frac{1}{f'(x_{2n})}. \tag{55}$$

for iterations (32a) and (32b), respectively.

Remark 1. Note that in general cases, it is difficult to find $M_2 = \max_{x\in[a,b]}|f''(x)|$. Therefore, instead of (54), (55) we can use the following parameters

$$\tau_n = \frac{1-\sqrt{1-2\tilde{a}_{2n}}}{\tilde{a}_{2n}}\bigg|_{n\to\infty} \to 1, \quad \omega_n = \frac{1}{f'(x_{2n+1})}, \tag{56}$$

$$\tau_n = 1, \quad \omega_n = \frac{1-\sqrt{1-2\tilde{a}_{2n}}}{\tilde{a}_{2n}f'(x_{2n})}\bigg|_{n\to\infty} \to \frac{1}{f'(x_{2n})}, \tag{57}$$

respectively. Here

$$\tilde{a}_{2n} = \left|\frac{f''(x_{2n})f(x_{2n})}{(f'(x_{2n}))^2}\right| \leqslant a_{2n} \tag{58}$$

and as shown in [6], holds the inclusion

$$I_{2n} \subseteq \tilde{I}_{2n} = \left[\frac{1-\sqrt{1-2\tilde{a}_{2n}}}{\tilde{a}_{2n}}, \frac{-1+\sqrt{1+4\tilde{a}_{2n}}}{\tilde{a}_{2n}}\right) \subseteq [1,2). \tag{59}$$

5. Numerical results

We consider two examples [8].

Example 1. $f(x) = \exp(x) - 4x^2 = 0$. It is easy to show that

(a) $f'(x) < 0, \quad f''(x) < 0 \quad$ at $x \in \left[\frac{1}{2}, 1\right]$, and $x^* \in \left(\frac{1}{2}, 1\right)$,
(b) $f'(x) > 0, \quad f''(x) < 0 \quad$ at $x \in \left[-\frac{1}{2}, 0\right]$, and $x^* \in \left(-\frac{1}{2}, 0\right)$.

Example 2. $f(x) = x^2 - 2\cos(x) = 0$. It is also easy to show that

(a) $f'(x) > 0, \quad f''(x) > 0 \quad$ at $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, and $x^* \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$,
(b) $f'(x) < 0, \quad f''(x) > 0 \quad$ at $x \in \left[-\frac{\pi}{2}, -\frac{\pi}{6}\right]$, and $x^* \in \left(-\frac{\pi}{2}, -\frac{\pi}{6}\right)$.

One can see that Theorem 1 and Corollary 1 are applicable in case of (a) of two examples, while Theorem 2 and Corollary 2 are in case of (b). The iteration was terminated by stopping criteria

$$|x_{2n+2} - x_{2n+1}| \leqslant \varepsilon = 10^{-15}. \tag{60}$$

The numerical results presented in Tables 1–3 confirm the theoretical behavior of convergence.

Table 1
Numerical results for Example 1(a).

n	$\tau_n = \left(1-\sqrt{1-2a_{2n}}\right)/a_{2n}, \quad \omega_n = 1/f'(x_{2n+1})$		$\tau_n = \left(1-\sqrt{1-2a_{2n}}\right)/a_{2n}, \quad \omega_n = 1/f'(1/2)$	
	x_{2n}	x_{2n+1}	x_{2n}	x_{2n+1}
0	1.000000000000000	0.705008413252650	1.000000000000000	0.705008413252650
1	0.714885141753139	0.714805912025241	0.720198556664536	0.714804319037903
2	0.714805912362778	0.714805912362778	0.714806809136289	0.714805912362735
3	0.714805912362778		0.714805912362802	0.714805912362778
4			0.714805912362778	

Table 2
Numerical results for Example 2(a).

n	$\tau_n = \left(1-\sqrt{1-2a_{2n}}\right)/a_{2n}, \quad \omega_n = 1/f'(x_{2n+1})$		$\tau_n = \left(1-\sqrt{1-2a_{2n}}\right)/a_{2n}, \quad \omega_n = 1/f'(\pi/6)$	
	x_{2n}	x_{2n+1}	x_{2n}	x_{2n+1}
0	1.570796326794897	0.951886943598052	1.570796326794897	0.951886943598052
1	1.023842847967236	1.021689527032909	1.076059433807942	1.021390754913898
2	1.021689954092259	1.021689954092185	1.021938659981420	1.021689948412844
3	1.021689954092185		1.021689958814336	1.021689954092185
4			1.021689954092185	

Table 3
Numerical results for Example 1(b) and Example 2(b).

n	$\tau_n = 1, \quad \omega_n = \left(1-\sqrt{1-2a_{2n}}\right)/(a_{2n}f'(x_{2n}))$			
	Example 1(b)		Example 2(b)	
	x_{2n}	x_{2n+1}	x_{2n}	x_{2n+1}
0	−0.500000000000000	−0.407756031328745	-1.570796326794897	−0.951886943598052
1	−0.407776709803781	−0.407776709404480	-1.023842847967236	−1.021689527032909
2	−0.407776709404480		-1.021689954092259	−1.021689954092185
3			-1.021689954092185	

6. Conclusions

One of the advantages of our iterations is that they give two-sided approximations of the solutions which allow one to control the error at each iteration step. It has been shown that the suitable iteration parameter allows us to control not only the convergence order, but also the convergence behavior. Numerical examples are given to illustrate the theoretical results.

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