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Higher-order accurate numerical solution of unsteady Burgers' equation

T. Zhanlav [a], O. Chuluunbaatar [a,b], V. Ulziibayar [c,*]

[a] Institute of Mathematics, National University of Mongolia, Mongolia
[b] Joint Institute for Nuclear Research, Dubna, 141980 Moscow region, Russia
[c] School of Applied Sciences, Mongolian University of Science and Technology, Mongolia

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ABSTRACT

Higher-order accurate finite-difference schemes for solving the unsteady Burgers' equation which often arises in mathematical modeling used to solve problems in fluid dynamics are presented. The unsteady Burgers' equation belongs to a few nonlinear partial differential equations which has an exact solution, and it allows one to compare the numerical solution with the exact one, and the properties of different numerical methods. We propose an explicit finite-difference scheme for a numerical solution of the heat equation with Robin boundary conditions. It has a sixth-order approximation in the space variable, and a third-order approximation in the time variable. As an application, we developed numerical schemes for solving a numerical solution of Burgers' equation using the relationship between the heat and Burgers' equations. This scheme has up to sixth-order approximation in the space variables. The main advantage of our approach is transition to one-dimensional equation which essentially reduces the computation costs compared to other direct methods for solving the unsteady Burgers' equation. The numerical results of test examples are found in good agreement with exact solutions for a wide range of Reynolds number and confirm the approximation orders of the schemes proposed.

1. Introduction

One version of two-dimensional Burgers' equation is the unsteady Burgers' equation given by

$$\frac{\partial u}{\partial t}+u\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\quad (x,y)\in D=[a,b]\times[c,d],\quad t>0. \tag{1}$$

Here $\nu = 1/R > 0$ is an arbitrary number and R is the Reynold's number.

It is well known, that Burgers' equation is suited to modeling fluid flows because it incorporates directly the interaction between the non-linear convection processes and the diffusive viscous processes [1,2]. Consequently, it is one of the principle model equations used to test the accuracy of new numerical methods. During the past decade, use of high-order numerical algorithms in the context of finite-difference methods has become relatively popular in computational fluid dynamics. However, in some cases the high-order truncation error of typical methods may become very large and thus may not be

* Corresponding author.
E-mail addresses: tzhanlav@yahoo.com (T. Zhanlav), chuka@jinr.ru (O. Chuluunbaatar), v_ulzii@yahoo.com (V. Ulziibayar).

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neglected. This implies that attempting to employ high-order methods for solving equations exhibiting erratic, turbulent-lire solutions may fail to produce expected results. It must be keeping in mind.

There are different numerical methods for solving Eq. (1) with a corresponding initial condition and boundary conditions [3–8]. In Jiwari [3] an efficient numerical scheme based on Haar wavelets and the quasilinearization process is developed for solving a nonlinear 1D Burgers' equation with Dirichlet boundary conditions. The distributed approximating functional method is applied for solving the 1D Burgers' equation and the unsteady Burgers' equation [4]. In Duan and Liu [5] unsteady Burgers' equation with the Dirichlet boundary conditions is solved by the lattice Boltzmann method. The higher-order accurate two-point compact alternating direction implicit algorithm was introduced to solve the unsteady Burgers' equation. A comparison of the proposed scheme with the fourth-order DuFort–Frankel [6] scheme was constructed in terms of accuracy and computational efficiency. Kutluay and Yagmurlu [7] proposed the modified Bi-quintic B-spline basis functions and applied to the unsteady Burgers' equation using the Galerkin method to obtain its numerical solution. In our previous paper [8] the higher-order finite-different schemes for solving 1D Burgers' equation with the homogeneous Dirichlet boundary conditions are presented.

The aim of the present paper is to construct stable and higher-order accurate finite-difference schemes to solve a unsteady Burgers' equation with inhomogeneous Dirichlet boundary conditions. It is realized by the following four steps.

1. The reduction of the unsteady Burgers' equation to the 1D Burgers' equation using the properties of the required solution of the original equation.
2. To solve the obtained heat equation with Robin boundary conditions on the uniform grids of the spatial and time intervals by means of an explicit finite-difference scheme. This scheme has a sixth-order approximation in the space variable, and a third-order approximation in the time variable, except boundary points of the spatial variable. We additionally used the fourth/sixth-order finite-difference approximations for the Robin boundary conditions.
3. To find a numerical solution of the 1D Burgers' equation by means of the numerical solution calculated in previous step of the heat equation. The obtained numerical solution has the same orders approximations in the space and time variables as numerical solution of the heat equation.
4. To find a numerical solution of the unsteady Burgers' equation by means of the numerical solution calculated in previous step of the 1D Burgers' equation.

As a test desk, the higher-order finite-difference schemes proposed are applied to the calculation of the several exact solvable examples. The numerical results are found in good agreement with exact solutions for a wide rang of the Reynolds number and confirm the approximation orders of the schemes proposed.

The structure of the paper is as follows. In Section 2, we present reductions of the unsteady Burgers' equation to the 1D Burgers' equation, and the obtained one to the heat equation. The higher-order accurate finite-difference schemes for solving the heat equation is presented in Section 3. The construction of the higher-order accurate finite-difference schemes for solution of the 1D Burgers' equation is given in Section 4. Numerical results are discussed in Section 5.

2. Reduction of the unsteady Burgers' equation to the 1D Burgers' equation

It is easy to show that by linear transformation of independent variables

$$z = x + y, \quad s = x - y, \quad \bar{t} = 2t, \tag{2}$$

the Eq. (1) is reduced to the following equation:

$$\frac{\partial u}{\partial \bar{t}} + u\frac{\partial u}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial s^2}\right). \tag{3}$$

Note that the rectangular region D by the transformation (2) leads to interval $A = a + c \leqslant z \leqslant b + d = B$.

If the solution $u(z, s, t)$ depends only on variables s and t, the Eq. (3) leads to the heat equation

$$\frac{\partial u}{\partial \bar{t}} = \nu\frac{\partial^2 u}{\partial s^2}. \tag{4}$$

Also, if the solution $u(z, s, \bar{t})$ depends only on variables z and $\bar{t}$, i.e, $u(z, s, t) \equiv u(z, \bar{t})$, the Eq. (3) is reduced to the 1D Burgers' equation

$$\frac{\partial u}{\partial \bar{t}} + u\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2}, \tag{5}$$

with an initial condition

$$u(z, 0) = g(z, 0), \quad z \in (A, B). \tag{6}$$

The boundary conditions for Eq. (5) are defined by

$$u(A, \bar{t}) = g(A, \bar{t}), \quad u(B, \bar{t}) = g(B, \bar{t}). \tag{7}$$

Table 1
The maximum absolute error $\|e\|_{\infty,h}$ and the Runge coefficient r_h. The factor x in the brackets denotes 10^x.

Example I	N	Fourth-order finite-difference scheme		Sixth-order finite-difference scheme	
		$\|e\|_{\infty,h}$	r_h	$\|e\|_{\infty,h}$	r_h
$v=1$	20	6.81630(-07)		1.09301(-09)	
$T=1/\sqrt{15}$	40	4.60638(-08)	14.80	1.89602(-11)	57.65
	80	2.99137(-09)	15.40	3.11914(-13)	60.79
	160	1.90541(-10)	15.70	4.99992(-15)	62.38
	320	1.20218(-11)	15.85	7.91258(-17)	63.19
	640	7.54910(-13)	15.92	1.24423(-18)	63.59
	1280	4.72930(-14)	15.96	1.95030(-20)	63.79
$v=0.1$	20	4.09922(-03)		4.22744(-04)	
$T=1/\sqrt{15}$	40	4.03057(-04)	10.17	1.33297(-05)	31.71
	80	3.15596(-05)	12.77	2.94802(-07)	45.21
	160	2.20669(-06)	14.30	5.47935(-09)	53.80
	320	1.45857(-07)	15.13	9.33799(-11)	58.67
	640	9.37449(-09)	15.56	1.52382(-12)	61.28
	1280	5.94147(-10)	15.78	2.43325(-14)	62.62
$v=0.01$	160	1.92093(-01)		2.93743(-02)	
$T=10/\sqrt{15}$	320	1.85362(-02)	10.36	9.64375(-04)	30.45
	640	1.48557(-03)	12.48	2.23559(-05)	43.13
	1280	1.05437(-04)	14.09	4.27027(-07)	52.35
	2560	7.03016(-06)	14.99	7.38644(-09)	57.81

There are many solutions of Eq. (1) that depend only on z and t [5–7,14,15]. As examples, we present here some of these solutions:

$$\text{Example I: } u(x,y,t)=\left(1+\exp\left(\frac{x+y-t}{2v}\right)\right)^{-1},$$
$$\text{Example II: } u(x,y,t)=\frac{1}{2}-\tanh\left(\frac{x+y-t}{2v}\right),$$
$$\text{Example III: } u(x,y,t)=\frac{1}{2}-\coth\left(\frac{x+y+1-t}{2v}\right).$$

The main advantage of our approach is to reduce the two-dimensional unsteady Burgers' equation to one-dimensional one. This allows us to use known higher accurate numerical schemes and to save computing time and memory of computers as compared to other direct methods [5–7] for solving two-dimensional unsteady Burgers' Eq. (1).

Here and below we consider only Eqs. (5)–(7). It is well known that, by the Hopf-Cole transformation

$$u(z,\bar{t})=-2v\frac{1}{\theta(z,\bar{t})}\frac{\partial\theta(z,\bar{t})}{\partial z}, \tag{8}$$

Burgers' Eq. (5) is reduced to the heat equation

$$\frac{\partial\theta(z,\bar{t})}{\partial\bar{t}}=v\frac{\partial^2\theta(z,\bar{t})}{\partial z^2}. \tag{9}$$

By (8) the initial condition (6) and boundary conditions (7) lead to

$$\theta(z,0)=\exp\left(-\frac{1}{2v}\int_A^z g(\xi,0)d\xi\right), \tag{10}$$

and

$$\frac{\partial\theta(z,\bar{t})}{\partial z}+\frac{1}{2v}g(z,\bar{t})\,\theta(z,\bar{t})=0 \quad \text{at} \quad z=A,B, \tag{11}$$

respectively. Thus, the Eqs. (5)–(7) are fully converted to problem (9)–(11).

3. Numerical solution of the heat equation

We suppose that the solution of the heat problem defined by Eqs. (9)–(11) is a sufficiently smooth function with respect to z and $\bar{t}$. The heat problem can be solved by a well-known Crank–Nicolson scheme [9] and more accurate explicit schemes proposed by Zhanlav in [10]:

$$\theta_i^n = \frac{\beta-\gamma}{\beta+\gamma}\theta_i^{n-2} + \frac{\beta\gamma}{\beta+\gamma}\left(\theta_{i-1}^{n-1} - 2\theta_i^{n-1} + \theta_{i+1}^{n-1}\right) + \frac{2\gamma}{\beta+\gamma}\theta_i^{n-1},$$
$$\gamma = \frac{2\tau\nu}{h^2}, \quad h = \frac{B-A}{N}, \quad i = 1,\ldots,N-1, \quad n = 2,3,\ldots \tag{12}$$

Here and throughout the work, θ_i^n is the approximate solution at the mesh points $(z_i = ih, \bar{t}_n = n\tau)$, where h is a spatial step, τ is a time step, and N is the number of partition of the interval $[A,B]$. In [10] it is shown that the scheme (12) is stable and its truncation error is of the order of $O(\tau^3 + h^6)$ provided that

$$\beta = 0.2, \quad \frac{\tau\nu}{h^2} = \frac{1}{\sqrt{60}} \quad \left(\text{or} \quad \gamma = \frac{1}{\sqrt{15}}\right). \tag{13}$$

When $\beta = 1$ the scheme (12) leads to the well-known DuFort–Frankel's one [9].

Note that the scheme (12) is used for the heat equation with Dirichlet boundary condition [10]. It is needed to adopt this scheme for Eq. (9) with Robin boundary conditions (11).

It should be mentioned that the scheme (12) is a three-level one in time. Hence, in order to find θ_i^n at level two, it requires values θ_i^n at level $n = 0,1$, i.e., θ_i^0 and θ_i^1. Using the Taylor expansion of $\theta(z,\tau)$ at point $(z,0)$ and Eq. (9), we obtain

$$\theta(z,\tau) = \theta(z,0) + \nu\frac{\partial^2\theta(z,0)}{\partial z^2}\tau + \frac{\nu^2}{2}\frac{\partial^4\theta(z,0)}{\partial z^4}\tau^2 + \cdots \tag{14}$$

From the initial condition (10) and Taylor expansion (14), we find θ_i^1 with the accuracy $O(\tau^3)$

$$\theta_i^1 = \left(1 + \nu\tau F_1(z_i) + \frac{\nu^2\tau^2}{2}F_2(z_i)\right)\exp\left(-\frac{1}{2\nu}\int_A^{z_i} g(\xi,0)d\xi\right), \quad i = 0,\ldots,N, \tag{15}$$

where

$$F_1(z) = -\frac{g_z'(z,0)}{2\nu} + \frac{g^2(z,0)}{4\nu^2},$$
$$F_2(z) = -\frac{g_z'''(z,0)}{2\nu} + \frac{4g(z,0)g_z''(z,0) + 3(g_z'(z,0))^2}{4\nu^2} - \frac{3g^2(z,0)g_z'(z,0)}{4\nu^3} + \frac{g^4(z,0)}{16\nu^4}. \tag{16}$$

From the Robin boundary conditions (11) using the asymmetric fourth-order and sixth-order finite-difference approximations of the first spatial derivative [13]:

$$\frac{\partial\theta(z,\bar{t})}{\partial z}\Big|_{z=A} = \frac{-25\theta_0^n + 48\theta_1^n - 36\theta_2^n + 16\theta_3^n - 3\theta_4^n}{12h} + O(h^4),$$
$$\frac{\partial\theta(z,\bar{t})}{\partial z}\Big|_{z=B} = \frac{25\theta_N^n - 48\theta_{N-1}^n + 36\theta_{N-2}^n - 16\theta_{N-3}^n + 3\theta_{N-4}^n}{12h} + O(h^4),$$
$$\frac{\partial\theta(z,\bar{t})}{\partial z}\Big|_{z=A} = \frac{-147\theta_0^n + 360\theta_1^n - 450\theta_2^n + 400\theta_3^n - 225\theta_4^n + 72\theta_5^n - 10\theta_6^n}{60h} + O(h^6),$$
$$\frac{\partial\theta(z,\bar{t})}{\partial z}\Big|_{z=B} = \frac{147\theta_N^n - 360\theta_{N-1}^n + 450\theta_{N-2}^n - 400\theta_{N-3}^n + 225\theta_{N-4}^n - 72\theta_{N-5}^n + 10\theta_{N-6}^n}{60h} + O(h^6), \tag{17}$$

Table 2
The same as in Table 1, but for Example II.

Example II	N	Fourth-order finite-difference scheme		Sixth-order finite-difference scheme	
		$\lVert e\rVert_{\infty,h}$	r_h	$\lVert e\rVert_{\infty,h}$	r_h
$\nu = 1$	320	1.30462(-06)		1.85387(-09)	
$T = 1/\sqrt{15}$	640	8.44299(-08)	15.45	3.06840(-11)	60.41
	1280	5.36962(-09)	15.72	4.93466(-13)	62.18
	2560	3.38538(-10)	15.86	7.82248(-15)	63.08
	5120	2.12510(-11)	15.93	1.23111(-16)	63.53
	10240	1.33108(-12)	15.96	1.93057(-18)	63.76
$\nu = 0.01$	640	3.63989(-03)		1.14887(-04)	
$T = 1/\sqrt{15}$	1280	2.74537(-04)	13.25	2.41846(-06)	47.50
	2560	1.89142(-05)	14.51	4.40048(-08)	54.95
	5120	1.24096(-06)	15.24	7.42060(-10)	59.30
	10240	7.94748(-08)	15.61	1.20472(-11)	61.59

we find the fourth-order and sixth-order approximations of θ_0^n and θ_N^n

$$\theta_0^n = \frac{48\theta_1^n - 36\theta_2^n + 16\theta_3^n - 3\theta_4^n}{25 - \frac{6h}{\nu} g(A, \bar{t}_n)},$$
$$\theta_N^n = \frac{48\theta_{N-1}^n - 36\theta_{N-2}^n + 16\theta_{N-3}^n - 3\theta_{N-4}^n}{25 + \frac{6h}{\nu} g(B, \bar{t}_n)}, \tag{18}$$

and

$$\theta_0^n = \frac{360\theta_1^n - 450\theta_2^n + 400\theta_3^n - 225\theta_4^n + 72\theta_5^n - 10\theta_6^n}{147 - \frac{30h}{\nu} g(A, \bar{t}_n)},$$
$$\theta_N^n = \frac{360\theta_{N-1}^n - 450\theta_{N-2}^n + 400\theta_{N-3}^n - 225\theta_{N-4}^n + 72\theta_{N-5}^n - 10\theta_{N-6}^n}{147 + \frac{30h}{\nu} g(B, \bar{t}_n)}, \tag{19}$$

respectively. Thus, we find θ_i^n for $i = 0, \ldots, N$ by the formulas (12), (18) or (12), (19).

4. Higher-order accurate finite-difference schemes for solution of 1D Burgers' equation

The higher-order finite-difference schemes, which presented in our previous paper [8], are applied for solving the Eqs. (5)–(7). For convenience, we recall them shortly. First, we consider the fourth-order finite-difference scheme:

$$v_{i-1}^n + 4v_i^n + v_{i+1}^n = -\frac{6\nu}{h}\left(\theta_{i+1}^n - \theta_{i-1}^n\right) \tag{20}$$

with boundary conditions

$$v_0^n = \theta_0^n \, g(A, \bar{t}_n), \quad v_N^n = \theta_N^n \, g(B, \bar{t}_n). \tag{21}$$

Here $v_i^n = \theta_i^n w_i^n$ and $w_i^n \equiv w(z_i, \bar{t}_n)$ is an approximate solution of $u(z_i, \bar{t}_n)$. The last system has a unique solution set $(v_0^n, v_1^n, \ldots, v_N^n)$ since its coefficient matrix is diagonally dominant [11]. It means that the tridiagonal system (20) and (21) has a unique solution set $(w_0^n, w_1^n, \ldots, w_N^n)$ for each $n = 1, 2, \ldots$, and it can be solved by efficient elimination method [5].

The sixth-order finite-difference scheme [8] has the form

$$v_{i-2}^n - 16v_{i-1}^n - 60v_i^n - 16v_{i+1}^n + v_{i+2}^n = c_i^n,$$
$$c_i^n = -\frac{3\nu}{h}\left(-\theta_{i-2}^n + 32\theta_{i-1}^n - 32\theta_{i+1}^n + \theta_{i+2}^n\right), \quad i = 2, \ldots, N-2. \tag{22}$$

Of course, besides of (21), we need additionally two end conditions v_1^n and v_{N-1}^n in order to solve the system (22). The solution procedure of system (22), (21) is essentially simplified by using a Z-folding algorithm [12]. Namely, if we use notation

$$Z_i^n = v_{i-1}^n + a v_i^n + v_{i+1}^n, \tag{23}$$

then it is easy to show that the Eq. (22) can be re written as

$$Z_{i-1}^n + bZ_i^n + Z_{i+1}^n = c_i^n, \quad i = 2, \ldots, N-2 \tag{24}$$

under conditions

$$a = -8 \mp 3\sqrt{14}, \quad b = -8 \pm 3\sqrt{14}. \tag{25}$$

Table 3
The same as in Table 1, but for Example III.

Example III	N	Fourth-order finite-difference scheme		Sixth-order finite-difference scheme	
		$\Vert e\Vert_{\infty,h}$	r_h	$\Vert e\Vert_{\infty,h}$	r_h
$\nu = 1$	320	7.53462(-09)		1.58377(-12)	
$T = 1/\sqrt{15}$	640	4.74283(-10)	15.88	2.52478(-14)	62.72
	1280	2.97497(-11)	15.94	3.98712(-16)	63.32
	2560	1.86273(-12)	15.97	6.26406(-18)	63.65
	5120	1.16526(-13)	15.98	9.81480(-20)	63.82
	10240	7.28624(-15)	15.99	1.53570(-21)	63.91
$\nu = 0.01$	320	7.24049(-05)		1.37914(-06)	
$T = 1/\sqrt{15}$	640	4.55170(-06)	15.90	2.01708(-08)	68.37
	1280	2.85640(-07)	15.93	3.05365(-10)	66.05
	2560	1.78938(-08)	15.96	4.69822(-12)	64.99
	5120	1.11973(-09)	15.98	7.28515(-14)	64.49
	10240	7.00275(-11)	15.98	1.13399(-15)	64.24

It means that the solution of penta-diagonal system (22) leads to two three-diagonal systems (24) and (23), consequently both systems have a diagonally dominance [11]. Now, the required additionally two end conditions Z_1^n and Z_{N-1}^n, are obtained from (23) and (8) as

$$Z_1^n = -2\nu\big(\theta'_z(A,\bar{t}_n) + a\theta'_z(z_1,\bar{t}_n) + \theta'_z(z_2,\bar{t}_n)\big),$$
$$Z_{N-1}^n = -2\nu\big(\theta'_z(z_{N-2},\bar{t}_n) + a\theta'_z(z_{N-1},\bar{t}_n) + \theta'_z(B,\bar{t}_n)\big). \tag{26}$$

Using the Robin boundary conditions (11) and the asymmetric sixth-order finite-difference approximations of the first spatial derivative [13], we can find the needed terms $\theta'_z(A,\bar{t}_n), \theta'_z(z_1,\bar{t}_n), \theta'_z(z_2,\bar{t}_n), \theta'_z(z_{N-2},\bar{t}_n), \theta'_z(z_{N-1},\bar{t}_n)$ and $\theta'_z(B,\bar{t}_n)$:

$$\begin{aligned}
\theta'_z(A,\bar{t}_n) &= -\frac{1}{2\nu}g(A,\bar{t}_n)\,\theta_0^n,\\
\theta'_z(z_1,\bar{t}_n) &= \frac{-10\theta_0^n - 77\theta_1^n + 150\theta_2^n - 100\theta_3^n + 50\theta_4^n - 15\theta_5^n + 2\theta_6^n}{60h},\\
\theta'_z(z_2,\bar{t}_n) &= \frac{2\theta_0^n - 24\theta_1^n - 35\theta_2^n + 80\theta_3^n - 30\theta_4^n + 8\theta_5^n - \theta_6^n}{60h},\\
\theta'_z(z_{N-2},\bar{t}_n) &= \frac{\theta_{N-6}^n - 8\theta_{N-5}^n + 30\theta_{N-4}^n - 80\theta_{N-3}^n + 35\theta_{N-2}^n + 24\theta_{N-1}^n - 2\theta_N^n}{60h},\\
\theta'_z(z_{N-1},\bar{t}_n) &= \frac{-2\theta_{N-6}^n + 15\theta_{N-5}^n - 50\theta_{N-4}^n + 100\theta_{N-3}^n - 150\theta_{N-2}^n + 77\theta_{N-1}^n + 10\theta_N^n}{60h},\\
\theta'_z(B,\bar{t}_n) &= -\frac{1}{2\nu}g(B,\bar{t}_n)\,\theta_N^n.
\end{aligned} \tag{27}$$

Substituting (27) into (26), we find Z_1^n and Z_{N-1}^n with order $O(h^6)$. Hence, the system of Eqs. (24) and (26) is solved by the efficient elimination method [11]. After them, one can solve

$$v_{i-1}^n + av_i^n + v_{i+1}^n = Z_i^n, \quad i = 1,\ldots,N-1,$$
$$v_0^n = -2\nu\ \theta'_z(A,\bar{t}_n), \quad v_N^n = -2\nu\ \theta'_z(B,\bar{t}_n). \tag{28}$$

Thus, we obtain the numerical solution $w_i^n = v_i^n/\theta_i^n$ of Burgers' Eqs. (5)–(7) with a higher accuracy provided that the solution θ_i^n of heat Eqs. (9)–(11) is found with a higher accuracy. The approximate values of solution of Eq. (1) with initial condition (6) and boundary condition (7) are found by formula

$$u_{kj}^n = w_i^n, \quad i = 0,\ldots,N, \tag{29}$$

where $u_{kj}^n = u(x_k, y_j, t_n)$, $x_k + y_j = z_i, a \leqslant x_k \leqslant b, c \leqslant y_j \leqslant d$ and $t_n = \bar{t}_n/2$.

5. Numerical results and discussion

In this section we demonstrate the accuracy of the fourth-order and sixth-order finite-difference schemes proposed, respectively, by solving three exact solvable unsteady Burgers' Eq. (1) and compare the numerical results $w(z,T)$ with the exact results $u(z,T)$. The maximum absolute error of the solution is defined by

$$\|e\|_{\infty,h} = \max_{0\leqslant i\leqslant N}|w(z_i,T) - u(z_i,T)|. \tag{30}$$

Convergence of the proposed schemes is reviewed by computation of the Runge coefficient

$$r_h = \frac{\|e\|_{\infty,h}}{\|e\|_{\infty,h/2}}. \tag{31}$$

The all computations are performed using a Fortran program with quadruple-precision arithmetics.

We consider the following exact solutions of Eq. (5):

$$u(z,\bar{t}) = \begin{cases} \left(1 + \exp\left(\frac{z}{2\nu} - \frac{\bar{t}}{4\nu}\right)\right)^{-1} & \text{for Example I},\\ \frac{1}{2} - \tanh\left(\frac{z}{2\nu} - \frac{\bar{t}}{4\nu}\right) & \text{for Example II},\\ \frac{1}{2} - \coth\left(\frac{z+1}{2\nu} - \frac{\bar{t}}{4\nu}\right) & \text{for Example III}.\end{cases} \tag{32}$$

Note that the second and third solutions called the kink and travelling wave solutions of Burgers' equation [15], respectively. In all the examples the initial and boundary conditions are taken from the exact solutions. It is easy to show that the corresponding heat problems (9)–(11) are also exact solvable, and normalized exact solutions with condition $\theta(0,0) = 1$ can be expressed as

$$\theta(z,\bar{t}) = \begin{cases} \frac{1}{2}\left(1+\exp\left(-\frac{z}{2\nu}+\frac{\bar{t}}{4\nu}\right)\right) & \text{for Example I}, \\ \exp\left(-\frac{z}{4\nu}+\frac{5\bar{t}}{16\nu}\right)\cosh\left(\frac{z}{2\nu}-\frac{\bar{t}}{4\nu}\right) & \text{for Example II}, \\ \exp\left(-\frac{z}{4\nu}+\frac{5\bar{t}}{16\nu}\right)\frac{\sinh\left(\frac{z+1}{2\nu}-\frac{\bar{t}}{4\nu}\right)}{\sinh\left(\frac{1}{2\nu}\right)} & \text{for Example III}. \end{cases} \quad (33)$$

The maximum absolute errors $\|e\|_{\infty,h}$ and the Runge coefficients r_h for the Examples I–III are presented in Tables 1–3. They are consistent with the theoretical expectations of $O(h^4)$ and $O(h^6)$. Also we see that for small number ν required large N to obtain higher accurate numerical solutions. Note that from Tables 1–3 we observed a slow convergence of the Runge coefficients r_h to the theoretical expectations (especially at small values of ν). This fact is a consequence of the application of the asymmetric approximations (18), (19) and (27) in a vicinity of the boundary points and due to the presence of large gradient in the solution. The results of numerical experiments demonstrate the expected accuracy and convergence order of proposed schemes.

6. Conclusions

The proposed higher-order finite-difference schemes are easy for implementation and can be used for a numerical solution of unsteady Burgers' equation with higher accuracy. The main advantage of the our schemes considered is reduced the two-dimensional unsteady Burgers' equation to one-dimensional Burgers equation that allows us to used known higher accurate numerical method. And thereby to save computing time and memory of computer as compared to the numerical methods for calculation of the two-dimensional Burgers' equation. The numerical results obtained demonstrated the accuracy and efficiency of the schemes considered. Also the numerical results show that the variation in the values of the Reynolds number does not adversely affect the numerical solutions. Since all numerical results obtained by the above methods show a reasonably good agreement with the exact one for modest values of ν, and also exhibit the expected convergence as the mesh size is decreased, the proposed methods can be considered to be competitive and worth recommendation.

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