UDC 519.6

On Newton-Type Methods with Fourth and Fifth-Order Convergence

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In this paper, we suggest and analyze new three-step iterative methods for solving nonlinear equations. The analysis of convergence shows that the proposed methods are fourth and fifth-order convergence. Several numerical examples are given to illustrate the efficiency and performance of the proposed methods. Comparison of different methods is also given.

Key words and phrases: iterative methods, order of convergence, Newton-type method, nonlinear equations.

1. Introduction

In recent years, much attention has been given to the development of new higher-order methods [1–12] for solving a nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f(x) : D \subseteq R \to R$ is a scalar or vector function for an open interval D. Among them the research of the high-order methods free from second derivatives is getting very active now. In [13] a fourth-order convergent two-step iterative method was suggested:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}, \quad n = 0, 1, \ldots \tag{2}$$

Here we assume that x^* is a simple root of Eq. (1) and x_0 is an initial guess sufficiently close to x^*. This paper can be considered as a continuation of our previous work [13] and we propose here two three-step iterative methods:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \, z_n = y_n - \frac{f(y_n)}{f'(x_n)}, \, x_{n+1} = y_n - \frac{f(y_n) + f(z_n)}{f'(x_n)}, \, n = 0, 1, \ldots, \tag{3}$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \, z_n = y_n - \frac{f(y_n)}{f'(y_n)}, \, x_{n+1} = y_n - \frac{f(y_n) + f(z_n)}{f'(y_n)}, \, n = 0, 1, \ldots, \tag{4}$$

with a fourth and fifth-order convergence, respectively. Of course, these methods belong to Newton-type ones and do not require a second derivative. It is worth to mention that the iterative method (3) looks more simple, especially for the system of Eq. (1) than (2), although they have the same order of convergence. In [2,3] three-step and fourth-order convergence iterative method was suggested using a decomposition technique. Unlike (2) and (3), this method requires a second order derivative, which is

Received 28th November, 2009.

The authors thank Prof. I.V. Puzynin, Prof. S.I. Vinitsky and Dr. A.A. Gusev for collaboration. O.C. acknowledges financial support from the RFBR grant 08-01-00604 and the theme 09-6-1060-2005/2010 "Mathematical support of experimental and theoretical studies conducted by JINR".

a serious drawback. Another fourth-order method free from derivatives was analyzed by M. Xiangjiang and W. Xinghua in [14].

2. Convergence Analysis

We now proceed to study the convergence analysis of the iterative method (4).

Theorem 1. *Let $x^* \in D$ be a simple root of a sufficiently smooth function $f(x) : D \subseteq R \to R$ for an open interval D. If x_0 is sufficiently close to x^*, then the three-step iterative method (4) has a fifth-order convergence.*

Proof. Let $e_n = x_n - x^*$ be the error at n-th iteration, and $a_n = \frac{f(x_n)}{f'(x_n)}$. Using Taylor series expansion of sufficiently smooth function $f(x)$ in the vicinity of x^* and x_n, easy shows that

$$a_n = e_n - \frac{1}{2}\frac{f''(x_n)}{f'(x_n)}e_n^2 + \frac{1}{6}\frac{f'''(x_n)}{f'(x_n)}e_n^3 - \frac{1}{24}\frac{f^{(iv)}(x_n)}{f'(x_n)}e_n^4 + O(e_n^5), \tag{5}$$

$$f(y_n) = \frac{1}{2}f''(x_n)a_n^2 - \frac{1}{6}f'''(x_n)a_n^3 + \frac{1}{24}f^{(iv)}(x_n)a_n^4 + O(a_n^5), \tag{6}$$

$$f'(y_n) = f'(x_n) - f''(x_n)a_n + \frac{1}{2}f'''(x_n)a_n^2 - \frac{1}{6}f^{(iv)}(x_n)a_n^3 + O(a_n^4), \tag{7}$$

$$f''(y_n) = f''(x_n) - f'''(x_n)a_n + \frac{1}{2}f^{(iv)}(x_n)a_n^2 + O(a_n^3). \tag{8}$$

From (6) and (7) we get

$$\omega_n \equiv \frac{f(y_n)}{f'(y_n)} = \frac{f''(x_n)}{2f'(x_n)}a_n^2 + b_n a^3 + c_n a_n^4 + O(a_n^5), \tag{9}$$

with

$$b_n = \frac{1}{f'(x_n)}\left(\frac{(f''(x_n))^2}{2f'(x_n)} - \frac{f'''(x_n)}{6}\right),$$

$$c_n = \frac{1}{f'(x_n)}\left(\frac{f^{(iv)}(x_n)}{24} + b_n f''(x_n) - \frac{f''(x_n)f'''(x_n)}{4f'(x_n)}\right).$$

Substituting (5) into (9) and after some calculations we obtain

$$\omega_n = \frac{f''(x_n)}{2f'(x_n)}e_n^2 - \frac{f'''(x_n)}{6f'(x_n)}e_n^3 + \left[\frac{f^{(iv)}(x_n)}{24f'(x_n)} - \frac{1}{8}\left(\frac{f''(x_n)}{f'(x_n)}\right)^3\right]e_n^4 + O(e_n^5). \tag{10}$$

In a similar way, we find that

$$\frac{f''(y_n)}{f'(y_n)} = \frac{f''(x_n)}{f'(x_n)} + \frac{1}{f'(x_n)}\left(\frac{(f''(x_n))^2}{f'(x_n)} - f'''(x_n)\right)a + O(a^2). \tag{11}$$

Also from (4) and (9) we get

$$f(z_n) = \frac{f''(y_n)}{2}\omega_n^2 + O(\omega_n^3). \tag{12}$$

From (5), (8), (10), (11) and (12) it follows that

$$\frac{f(z_n)}{f'(y_n)} = \frac{1}{8}\left(\frac{f''(x_n)}{f'(x_n)}\right)^3 e_n^4 + O(e_n^5). \tag{13}$$

From (4), we have

$$e_{n+1} = e_n - a - \frac{f(y_n) + f(z_n)}{f'(y_n)}. \tag{14}$$

Substituting (5), (10) and (13) into (14), we have $e_{n+1} = O(e_n^5)$, i.e., the iterative method (4) has a fifth-order convergence in the vicinity of the root x^*. □

For the convergence order of method (3) we have the following theorem:

Theorem 2. *Let $x^* \in D$ be a simple root of a sufficiently smooth function $f : D \subseteq R \to R$ for an open interval D. If x_0 is sufficiently close to x^*, then the three-step iterative method (3) has a fourth-order convergence.*

Proof. Let $e_n = x_n - x^*$ be the error at n-th iteration, $d_n = y_n - x_n = -\frac{f(x_n)}{f'(x_n)}$, $k_n = z_n - y_n = -\frac{f(y_n)}{f'(x_n)}$ and $s_n = z_n - x_n = d_n + k_n$. Using Taylor series expansion of sufficiently smooth function $f(x_n)$ in the vicinity of x^* and x_n, we have

$$\begin{aligned} f(x_n) &= f'(x_n)e_n - \frac{1}{2}f''(x_n)e_n^2 + \frac{1}{6}f'''(x_n)e_n^3 - \frac{1}{24}f^{(iv)}(x_n)e_n^4 + O(e_n^5), \\ f(y_n) &= \frac{1}{2}f''(x_n)d_n^2 + \frac{1}{6}f'''(x_n)d_n^3 + \frac{1}{24}f^{(iv)}(x_n)d_n^4 + O(d_n^5), \\ f(z_n) &= f(x_n) + f'(x_n)s_n + \frac{1}{2}f''(x_n)s_n^2 + \frac{1}{6}f'''(x_n)s_n^3 + \frac{1}{24}f^{(iv)}(x_n)s_n^4 + O(s_n^5). \end{aligned} \tag{15}$$

Also we can easy show that

$$\begin{aligned} d_n &= -e_n + \frac{1}{2}\frac{f''(x_n)}{f'(x_n)}e_n^2 - \frac{1}{6}\frac{f'''(x_n)}{f'(x_n)}e_n^3 + \frac{1}{24}\frac{f^{(iv)}(x_n)}{f'(x_n)}e_n^4 + O(e_n^5), \\ k_n &= -\frac{1}{2}\frac{f''(x_n)}{f'(x_n)}e_n^2 + \left(\frac{1}{6}\frac{f'''(x_n)}{f'(x_n)} + \frac{1}{2}\left(\frac{f''(x_n)}{f'(x_n}\right)^2\right)e_n^3 - \frac{B_n}{f'(x_n)}e_n^4 + O(e_n^5), \\ s_n &= -e_n + \frac{1}{2}\left(\frac{f''(x_n)}{f'(x_n)}\right)^2 e_n^3 + \left(\frac{1}{24}\frac{f^{(iv)}(x_n)}{f'(x_n)} - \frac{B_n}{f'(x_n)}\right)e_n^4 + O(e_n^5), \end{aligned} \tag{16}$$

where

$$B_n = \frac{1}{24}f^{(iv)}(x_n) + \frac{1}{2}f''(x_n)\left(\frac{1}{4}\left(\frac{f''(x_n)}{f'(x_n)}\right)^2 + \frac{1}{3}\frac{f'''(x_n)}{f'(x_n)}\right) + \frac{1}{4}\frac{f'''(x_n)f''(x_n)}{f'(x_n)}.$$

From (3) we obtain

$$e_{n+1} = e_n + s_n - \frac{f(z_n)}{f'(x_n)},$$

and after some calculation using (15), (16), we get $e_{n+1} = \frac{1}{2}\left(\frac{f''(x_n)}{f'(x_n)}\right)^2 e_n^4 + O(e_n^5)$, i.e., the iterative method (3) has fourth-order convergence in the vicinity of the root x^*. □

Remark 1. The proposed iterative methods (3) and (4) work well for solving a system of nonlinear equations of kind (1). Implementation of these methods looks like

$$\begin{aligned} f'(x_n)\nu_{1n} &= -f(x_n), \quad y_n = x_n + \nu_{1n}, \\ f'(x_n)\nu_{2n} &= -f(y_n), \quad z_n = y_n + \nu_{2n}, \\ f'(x_n)\nu_{3n} &= -\left(f(y_n) + f(z_n)\right), \quad x_{n+1} = y_{n+1} + \nu_{3n}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} f'(x_n)\nu_{1n} &= -f(x_n), \quad y_n = x_n + \nu_{1n}, \\ f'(y_n)\nu_{2n} &= -f(y_n), \quad z_n = y_n + \nu_{2n}, \\ f'(y_n)\nu_{3n} &= -\left(f(y_n) + f(z_n)\right), \quad x_{n+1} = y_{n+1} + \nu_{3n}. \end{aligned} \tag{18}$$

respectively. The algorithms (17) and (18) show that it is sufficient to invert the Jacobian matrices $f'(x_n)$ and $f'(y_n)$ only ones and twice, respectively, thus reducing the CPU time.

3. Numerical Examples and Efficiency Analysis

We consider the definition of the efficiency index [8] as $p^{1/w}$, where p is the order of the method and w is the number of the function evaluations per iterations required by the method, which is counted as a sum of the number of evaluations of the $f(x)$ itself plus the number of evaluations of the first derivative $f'(x)$. A theoretical comparison of efficiency index of various methods for the scalar equation (1) is given in Table 1. The comparison is given for one-, two- and three-step methods:

NM – Newton's method
MNM – modification of Newton's method [8]
MHM – modified Halley's method [9]
Alg 2.2, 2.3 – algorithms 2.2 and 2.3 in [2]

with our methods (2), (3) and (4).

Since $\sqrt[4]{3} < \sqrt[5]{5} < \sqrt{2} < \sqrt[3]{3} < \sqrt[3]{4}$ or
$1.3161 < 1.3797 < 1.4142 < 1.4422 < 1.5874$,

the efficiency index of these methods does not differ from one another essentially. The efficiency index of methods (2) and (3) is equal to those for Newton's method and it is compatible with those for MNM and MHM.

Table 1

Theoretical comparison of efficiency index of methods

Steps	1	2					3		
Methods	NM	MNM	MHM	Alg 2.2	(2)	[14]	Alg 2.3	(3)	(4)
p	2	3	3	3	4	4	3	4	5
w	2	3	3	3	4	3	4	4	5
$p^{1/w}$	1.414	1.442	1.442	1.442	1.414	1.587	1.316	1.414	1.380

Now we present some examples to illustrate the efficiency of the proposed methods (2), (3) and (4) and compare them with Newton's and modified Newton's methods and then fourth-order convergence method given by [14] for various scalar function equations. All computations are carried out with a double arithmetic precision and the number of iterations n such that $|f(x_n)| \leqslant 1.0e-16$ is tabulated (see Table 2). From Table 2 one can see that the fourth-order convergence methods [14], (2) and (3), as well as fifth-order convergence method (4) are compatible.

Now we can test these methods for a nonlinear system of equations:

$$F(x) \equiv \begin{pmatrix} x_1x_3 + x_2x_4 + x_3x_5 + x_4x_6 \\ x_1x_5 + x_2x_6 \\ x_1 + x_3 + x_5 - 1 \\ -x_1 + x_2 - x_3 + x_4 - x_5 + x_6 \\ -3x_1 - 2x_2 - x_3 + x_5 + 2x_6 \\ 3x_1 - 2x_2 + x_3 - x_5 + 2x_6 \end{pmatrix} = 0,$$

derived for constructing an orthonormal wavelet system with compact support [15]. Various iterative methods have been used for solving this system starting with initial

Table 2

Comparison of various iterative methods

$f(x)$	x_0	MNM	[14]	(2)	(3)	(4)
$f_1(x) \equiv x^2 - e^x - 3x + 2$	3	5	3	3	3	3
$f_2(x) \equiv \cos(x) - x$	1	3	2	2	2	2
$f_3(x) \equiv (x-1)^3 - 1$	2.5	4	3	3	3	3
$f_4(x) \equiv x^3 - 10$	1.5	5	3	3	4	3
$f_5(x) \equiv xe^{x^2} - \sin^2 x + 3\cos x + 5$	-2	6	4	4	5	4
$f_6(x) \equiv e^{x^2+7x-30} - 1$	5.5	26	19	22	27	19

guess $x^{(0)} = (0,0,0,1,1,0)^T$. The number of iterations n such that $\|F(x)\| \leqslant 1.0e-16$ and the number of correct digits after a decimal point of solution to this system are shown in Tables 3 and 4.

Table 3

Number of iterations

NM	Alg 2.3	MNM	(2)	(3)	(4)
7	6	4	4	3	3

Table 4

Number of correct digits after decimal point

	MN	Alg 2.3	MNM	(2)	(3)	(4)
$m = 1$	0	0	0	0	0	1
$m = 2$	0	0	2	4	4	9
$m = 3$	2	2	6	18	16	59

m is number of iterations

From Tables 3 and 4 we see that the fifth-order convergent method (4) yields a more precise result in a less number of iterations as compared to the fourth-order methods (2) and (3) especially in the case when used a higher arithmetic precision. The last three calculations in Table 4 was performed using MAPLE.

4. Conclusions

We have suggested and analyzed new three-step Newton-type methods with a free second derivative for solving nonlinear equations with a fourth and fifth-order convergence. We have obtained new iteration methods of Newton-type. From **theorems 1** and **2**, we have proved that the order of convergence of the new methods (2), (3) and (4) is four and five, respectively. From the numerical examples we see that the new methods have a great practical utility especially in the case of higher accuracy calculations.

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УДК 519.6

О методах Ньютоновского типа со сходимостью четвертого и пятого порядка

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В работе предложены и анализируются новые трехшаговые итерационные методы решения нелинейных уравнений. Анализ сходимости показывает, что предложенные методы являются сходимостью четвёртого и пятого порядка. Чтобы проиллюстрировать эффективность предложенных методов, приводятся несколько численных примеров. Также проводится сравнение различных методов.

Ключевые слова: итерационные методы, порядок сходимости, методы ньютоновского типа, нелинейные уравнения.