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КРАТКОЕ ОПИСАНИЕ ДВУСТОРОННЕГО ПРИБЛИЖЕНИЯ НЕКОТОРЫХ МЕТОДОВ НЬЮТОНОВСКОГО ТИПА

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Предложен и проанализирован метод, являющийся комбинацией демпфированного метода Ньютона и простого метода Ньютона. Показано, что он дает двустороннее приближение к точному решению, которое может быть использовано для апостериорной оценки. Эффективность предложенного метода демонстрируется на численных примерах.

A BRIEF DESCRIPTION OF TWO-SIDED APPROXIMATION FOR SOME NEWTON'S TYPE METHODS

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We suggest and analyze a combination of a damped Newton's method and a simplified version of Newton's one. We show that the proposed iterations give two-sided approximations of the solution which can be efficiently used as posteriori estimations. Some numerical examples illustrate the efficiency and performance of the method proposed.

1. Introduction

In the last decade, new iterative methods containing parameters for a numerical solving of nonlinear equations have been developed by many authors. The role of these parameters play, for example, a damped parameter in Newton type methods [1-6], interpolation nodes in inverse polynomial interpolation methods [7,8]. They can be controlled not only by the convergence order, but also by the convergence behavior. One of the advantages of such methods is that they give two-sided approximations of the solutions which allow one to control the error at each iteration step [3,6,8]. In this paper we will consider a combination of the damped Newton's method and the simplified Newton method. The extended version of this paper will be published elsewhere.

Let $a,b\in\mathbb{R},\ a<b,\ f:[a,b]\to\mathbb{R}$ and consider the following nonlinear equation

$$f(x)=0. \tag{1}$$

Assume that $f(x)\in C^3[a,b]$, $f'(x)\neq 0$, $x\in[a,b]$ and Eq. (1) has a unique root $x^*\in[a,b]$. For a numerical solution of (1) we propose the following iterations

$$x_{2n+1}=x_{2n}-\tau_n\frac{f\left(x_{2n}\right)}{f'\left(x_{2n}\right)},\quad n=0,1,..., \tag{2a}$$

$$x_{2n+2}=x_{2n+1}-\omega_n f\left(x_{2n+1}\right). \tag{2b}$$

Here $\tau_n>0$ and ω_n are the iteration parameters to be determined properly. It should be mentioned that the first iteration (2a) is a continuous analogy of Newton's method (or damped Newton's method), while the second ones (2b) is simple iterations. In [6] proved was a two-sided approximation behavior of iterations (2) with

$$\omega_n=\frac{1}{f'\left(x_{2n+1}\right)}, \tag{3}$$

and it was shown that the convergence rate of this iterations is 4 when $\tau_n\to 1$ as $n\to\infty$.

On the other hand, the iterations (2) can be considered as simple iterations

$$x_{2n+1}=p\left(x_{2n}\right),\quad x_{2n+2}=q\left(x_{2n+1}\right)=q\left(p\left(x_{2n}\right)\right),\quad n=0,1,..., \tag{4}$$

for two equations

$$x-p(x)=0,\quad x-q(x)=0, \tag{5}$$

which are equivalent to the above equation (1) and with functions

$$p\left(x\right)=x-\tau\frac{f\left(x\right)}{f'\left(x\right)},\quad q\left(x\right)=x-\omega f\left(x\right). \tag{6}$$

2. The convergence of the proposed iterations

Suppose that [6]

$$\left|\frac{f''(x)}{\left(f'(x)\right)^2}f\left(x\right)\right|\leq M_2\left|\frac{f\left(x\right)}{\left(f'(x)\right)^2}\right|\leq a\left(x\right)<\frac{4}{9},\qquad x\in\left[a,b\right], \tag{7}$$

where $M_2=\max\limits_{x\in[a,b]}\left|f''\left(x\right)\right|$. Then it is easy to show that the function $p(x)$ satisfies

$$0<p'\left(x_{2n}\right)<1,\qquad n=0,1,... \tag{8}$$

under condition

$$\tau_n \in \left]0, \frac{1}{1-a_{2n}}\right[, \qquad a_{2n} = M_2 \left|\frac{f(x_{2n})}{\left(f'(x_{2n})\right)^2}\right|. \tag{9}$$

A sufficient condition for $q(x)$ to be decreasing is

$$\omega_n f'(x_{2n+1}) > 1, \qquad n = 0,1,... \tag{10}$$

It should be noted that the conditions (8) and (10) were used first in [7,8] for bilateral approximations of Aitken-Steffensen-Hermite type methods.

Using Taylor expansion of $f(x_{2n+2})$ at point x_{2n+1}, and (2b) we obtain

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} = 1 - \omega_n f'(x_{2n+1}) + \frac{f''(\xi_{2n})}{2} f(x_{2n+1})\omega_n^2, \tag{11}$$

where $\xi_{2n} = \theta x_{2n+2} + (1-\theta)x_{2n+1}$, $\theta \in]0,1[$.

Lemma 1. *Suppose that*

$$f''(\xi_{2n}) f(x_{2n+1}) < 0, \quad n = 0,1, \tag{12}$$

and the inequality (10) holds. Then

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} < 0, \qquad n = 0,1,... \tag{13}$$

Proof. By (12) and (10) it follows from (11) that

$$\frac{f(x_{2n+2})}{f(x_{2n+1})} < 1 - \omega_n f'(x_{2n+1}) < 0. \tag{14}$$

The Lemma is proved. □

Analogously, using Taylor expansion of $f(x_{2n+1})$ at point x_{2n}, and (2a) we obtain

$$\frac{f(x_{2n+1})}{f(x_{2n})} = 1 - \tau_n + \frac{f''(\eta_n)}{2} \frac{f(x_{2n})}{\left(f'(x_{2n})\right)^2} \tau_n^2, \qquad n = 0,1,... \tag{15}$$

where $\eta_{2n} = \alpha x_{2n+1} + (1-\alpha)x_{2n}$, $\alpha \in]0,1[$.

Lemma 2. *Suppose that the inequality* (7) *holds. Then*

$$\frac{f(x_{2n+1})}{f(x_{2n})} < 0, \qquad n = 0,1,... \tag{16}$$

under condition

$$\tau_n \in l_{2n} = \left[\frac{1-\sqrt{1-2a_{2n}}}{a_{2n}}, \frac{-1+\sqrt{1+4a_{2n}}}{a_{2n}} \right[\subseteq [1,2[. \quad (17)$$

Proof. From (15) we obtain

$$\frac{f(x_{2n+1})}{f(x_{2n})} < 1-\tau_n + \frac{a_{2n}}{2}\tau_n^2 \le 0. \quad (18)$$

From this it follows (16) under condition

$$\tau_n \in \left[\frac{1-\sqrt{1-2a_{2n}}}{a_{2n}}, \frac{1+\sqrt{1-2a_{2n}}}{a_{2n}} \right]. \quad (19)$$

On the other hand, as shown in [5], the iteration parameter τ_n must be taken from the τ-region of convergence of iteration (2a)

$$\tau_n \in \left] 0, \frac{-1+\sqrt{1+4a_{2n}}}{a_{2n}} \right[\subseteq]0,2[. \quad (20)$$

Hence the inequality (16) is valid for $\tau_n \in I_{2n}$, and I_{2n} is not an empty interval because of (7). The Lemma is proved. □

Now we are ready to prove the following theorem when $f(x)$ is increasing and convex on the interval $x \in [a,b]$.

Theorem 1. *Let* $x_0 \in \left] x^*, b \right]$, *and* $f(x)$ *satisfies the following conditions:*

i_1) $f'(x) > 0, \quad x \in [a,b],$

ii_1) $f''(x) > 0, \quad x \in [a,b],$

iii_1) *the inequality* (7) *holds.*

If the parameters τ_n and ω_n are chosen such that

$$\tau_n \in \left] 0, \frac{1}{1-a_{2n}} \right[\cap l_{2n}, \quad n = 0,1,... \quad (21)$$

and

$$\omega_n f'(a) \ge 1, \quad n = 0,1,... \quad (22)$$

then the following relations hold:

j_1) $x_1 < x_3 < ... < x_{2n+1} < x^* < x_{2n} < ... < x_2 < x_0,$

jj_1) $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n} = x^*.$

Proof. By i_1) it follows that $x^* \in]a,b[$ is the unique solution of Eq. (1). From (8)–(10) the functions $p(x)$ and $q(x)$ have no extremum on the own domain of definition, and $p(a)>a$, $p(b)<b$, $q(a)>a$, $q(b)<b$ because of $f(a)<0$ and $f(b)>0$. Therefore, all the approximations generated by (2a) and (2b) belong to $[a,b]$, i.e. x_{2n+1}, $x_{2n+2} \in]a,b[$, $n=0,1,...$. By assumption of theorem $f(x_0)>0$. Then, according to Lemma 2, from (16) it follows that $f(x_1)<0$. By ii_1), $f'(x)$ is increasing on the interval $[a,b]$. According to (22), we have

$$\omega_0 \geq \frac{1}{f'(a)} > \frac{1}{f'(x)}, \qquad x \in]a,b[, \tag{23}$$

which holds, for instance, for x_1, i.e. the condition (10) is fulfilled for $n=0$. By virtue of ii_1) and $f(x_1)<0$ the assumption (12) is valid for $n=0$. Then, according to Lemma 1, from (13) we obtain $(x_2)>0$. By induction on n from (13) and (16) one can show that

$$f(x_{2n})>0, \quad f(x_{2n+1})<0, \tag{24}$$

and also can prove (8) and (10) for all $n=0,1,...$.

Thus the sequence $\{x_{2n+1}\}$ generated by (2) (or (4)) is increasing and the sequence $\{x_{2n+2}\}$ generated by (2b) (or (4)) is decreasing. Consequently we have

$$x_1 < x_3 < ... < x_{2n+1} < x^* < x_{2n} < ... < x_2 < x_0, \tag{25}$$

i.e. j_1) is proved. The jj_1) follows from j_1) passing to the limit $n\to\infty$. Since $f(x)$ is increasing function on the interval [a,b] then we have

$$f(x_1) < f(x_3) < ... < f(x_{2n+1}) < f(x^*) < f(x_{2n}) < ... < f(x_2) < f(x_0), \tag{26}$$

and

$$\lim_{n\to\infty} f(x_{2n+1}) = \lim_{n\to\infty} f(x_{2n}) = f(x^*) = 0. \tag{27}$$

It should be pointed out that the intersection of two intervals in (21) is not an empty set because of (7). The Theorem is proved. □

The remainder cases of behavior of $f(x)$ are studied in a similar way. Indeed we consider instead of Eq. (1) the auxiliary equation

$$g(y)=0, \tag{28}$$

where

$$g(y) = \begin{cases} -f(y), & y\in[a,b], \quad if\ f'(x)<0, \ f''(x)<0, \quad x\in[a,b], \\ -f(-y), & y\in[-b,-a], \quad if\ f'(x)>0, \ f''(x)<0, \ x\in[a,b], \\ f(-y), & y\in[-b,-a], \quad if\ f'(x)<0, \ f''(x)>0, \ x\in[a,b]. \end{cases} \tag{29}$$

The function $g(y)$ satisfies the conditions of Theorem 1.

Now we proceed to study of local convergence of proposed iterations (2).

Theorem 2. *Assume that* $f(x)\in C^4[a,b]$, $f'(x)\neq 0$, $x\in[a,b]$, *and there exists the unique solution* $x^*\in[a,b]$ *of Eq.* (1). *Then the q-convergence order of the sequence* $\{x_n\}$ *generated by iterations* (2) *is at least* 2, *when* $\tau_n\to 1$ *as* $n\to\infty$.

Proof. Let $e_n = x_n - x^*$. From (2a) and (2b) it follows that

$$e_{2n+1} = e_{2n} - \tau_n \frac{f(x_{2n}) - f(x^*)}{f'(x_{2n})}, \tag{30a}$$

$$e_{2n+2} = e_{2n+1} - \omega_n \left(f(x_{2n+1}) - f(x^*) \right). \tag{30b}$$

Using the Taylor expansions for $f(x^*)$ at points x_{2n} and x_{2n+1} in (30a) and (30b), respectively, we obtain following when $\tau_n \to 1$ as $n\to\infty$

$$\begin{aligned} &e_{2n+2} = O\left(e_{2n}^2\right), \quad \textit{if} \quad \omega_n = \text{const},\\ &e_{2n+2} = O\left(e_{2n}^3\right), \quad \textit{if} \quad \omega_n - \left(f'(x_{2n+1})\right)^{-1} = O(e_{2n}),\\ &e_{2n+2} = O\left(e_{2n}^4\right), \quad \textit{if} \quad \omega_n = \left(f'(x_{2n+1})\right)^{-1}. \end{aligned} \tag{31}$$

The Theorem is proved. □

From the Theorem 1–2 it is clear that the best choice of parameters are for iterations (2):

$$\tau_n = \left.\frac{1-\sqrt{1-2a_{2n}}}{a_{2n}}\right|_{n\to\infty} \to 1, \qquad \omega_n = \frac{1}{f'(x_{2n+1})}. \tag{32}$$

3. Numerical results

We considered $f(x) = \exp(x) - 2x^2 - x^3/3 = 0$. This equation has 4 roots It is easy to show that

$a) f'(x) > 0,\quad f''(x) > 0 \quad at \quad x\in[3.5, 4.5],$ *and* $x^* \in]3.5, 4.3[,$

$b) f'(x) < 0,\quad f''(x) < 0 \quad at \quad x\in[1, 1.5],$ *and* $x^* \in]1, 1.5[,$

$c) f'(x) > 0,\quad f''(x) < 0 \quad at \quad x\in[-1, 0],$ *and* $x^* \in]-1, 0[,$

$d) f'(x) > 0,\quad f''(x) < 0 \quad at \quad x\in[-7, -5],$ *and* $x^* \in]-7, -5[.$

Table 1. Numerical results.

n	Example a)	
	x_{2n}	x_{2n+1}
0	4.300000000000000	3.907141947701772
1	3.941963026936173	3.940806198327124
2	9.940806911126752	3.940806911126253
3	3.940806911126253	
n	Example b)	
	x_{2n}	x_{2n+1}
0	1.500000000000000	1.140241823567237
1	1.152335575731209	1.152252502154623
2	1.152252502332163	1.152252502332163
3	1.152252502332163	
n	Example c): $x_n = -y_n$	
	x_{2n}	x_{2n+1}
0	–1.000000000000000	–0.505411786074046
1	–0.562559445147446	–0.561019258063384
2	–0.561019587389929	–0.561019587389879
3	–0.561019587389879	
n	Example d): $x_n = -y_n$	
	x_{2n}	x_{2n+1}
0	–7.000000000000000	–5.969049117475682
1	–6.000113568662283	–5.999793371863974
2	–5.999793380403996	–5.999793380403996
3	–5.999793380403996	

The iteration was terminated by stopping criteria

$$\left|x_{2n+2} - x_{2n+1}\right| \le \varepsilon = 10^{-15}. \quad (33)$$

Numerical results presented in Table 1, confirm the theoretical behavior of convergence.

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