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CONVERGENCE OF A CONTINUOUS ANALOG OF NEWTON'S METHOD FOR SOLVING NONLINEAR EQUATIONS

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The influence of the parameter in the continuous analog of Newton's method (CANM) on the convergence and on the convergence rate is studied. A τ-region of convergence of CANM for both scalar equations and equations in a Banach space is obtained. Some almost optimal choices of the parameter are proposed. It is also shown that the well-known higher order convergent iterative methods lead to the CANM with an almost optimal parameter. Several sufficient convergence conditions for these methods are obtained.

Keywords: iterative methods, rate of convergence, Newton-type methods, nonlinear equations.

Introduction. In resent years much attention has been paid to the development of new high-order iterative methods for solving nonlinear equations [1–13]. Among them there are the methods obtained by combining Newton's method with other one-step methods [7, 8, 11, 12]. On the other hand, the so-called continuous analog of Newton's method (CANM), or the damped Newton method, with a parameter is often used [13, 14], although its convergence order is less than that of the above-mentioned methods. It is well known that a suitable choice of this parameter allows us not only to enlarge the domain of convergence, but also to control the convergence of the method in general. At present there are some choices of this parameter used in practice [13, 14]. In this paper we show that such suitable choices of the iteration parameter in CANM allow one even to speed up the convergence of the method under consideration.

1. Sufficient convergence conditions for CANM for scalar equations. We consider the nonlinear equation

$$f(x) = 0, \tag{1.1}$$

where $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a nonlinear twice continuously differentiable function on an open interval $\Omega_0 \subseteq \Omega$. We assume that x^* is a simple root of equation (1.1), i.e., $f(x^*) = 0$ and $f'(x^*) \neq 0$. The well-known CANM for equation (1.1) is

$$x_{n+1} = x_n - \tau_n \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \dots, \tag{1.2}$$

where x_0 is an initial guess and $\tau_n > 0$ is the iteration parameter. If $\tau_n \equiv 1$, then (1.2) leads to Newton's method.

Theorem 1. *We assume that*

(i) $|f''(x)| \leqslant k, \quad x \in \Omega_0,$

(ii) $\left|\big(f'(x_0)\big)^{-1}\right| \leqslant \beta,$

(iii) $\left|\big(f'(x_0)\big)^{-1} f(x_0)\right| \leqslant \eta, \quad a_0 = k\beta\eta,$

(iv) $k\left|\big(f'(x_n)\big)^{-1}\right| \left|\big(f'(x_n)\big)^{-1} f(x_n)\right| \leqslant a_n < 2, \quad n = 0, 1, \dots,$

and $\tau_n \in I_n = \left(0, \dfrac{-1 + \sqrt{1 + 4a_n}}{a_n}\right) \subseteq (0, 2)$. *Then, the sequence* $\{x_n\}$ *defined by* (1.2) *and starting at* $x_0 \in \Omega_0$ *converges to a solution* x^* *of equation* (1.1).

1.1. Some choices of the iteration parameter. We assume that $f(x) \in C^3(\Omega_0)$ and x_n is sufficiently close to x^*. Then, the Taylor expansion of $f(x)$ about x_n gives

$$f(x_{n+1}) = (1 - \tau_n) f(x_n) + \frac{f''(x_n)}{2} \left(\frac{f(x_n)}{f'(x_n)}\right)^2 \tau_n^2 + O\big(f^3(x_n)\big).$$

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From this it follows that

$$f^2(x_{n+1}) = \big[(1-\tau_n)^2 f(x_n)^2 + 2(1-\tau_n)\tau_n^2 f(x_n)A_n + A_n^2\tau_n^4\big] + O\big(f^4(x_n)\big), \tag{1.3}$$

where

$$A_n = \frac{f''(x_n)}{2}\left(\frac{f(x_n)}{f'(x_n)}\right)^2 \neq 0. \tag{1.4}$$

It is worth to mention that the remainder $O\big(f^4(x_n)\big)$ in (1.3) can be replaced by $O\big(f^5(x_n)\big)$ when $\tau_n \to 1$. In this sense we may call the quantity in the square brackets the main part of expansion (1.3). Our aim is to choose the iteration parameter to be a minimum point of the main part in (1.3), i.e.,

$$\varphi(\tau_n) \equiv 4A_n^2\tau_n^3 + 2A_n f(x_n)(2\tau_n - 3\tau_n^2) - 2(1-\tau_n)f^2(x_n) = 0. \tag{1.5}$$

Since $\varphi(0) = -2f(x_n)^2 < 0$ and $\varphi(2) = 2\big(4A_n - f(x_n)\big)^2 > 0$, there exists at least one root τ_n^* of equation (1.5) that belongs to $(0, 2)$. We call this root τ_n^* an optimal one. However, it is difficult to find τ_n^* by solving the cubic polynomial equation (1.5). To overcome this difficulty, as in [17], we use the notation

$$\theta_n = A_n\tau_n^2 \tag{1.6}$$

and rewrite (1.3) in the form

$$f^2(x_{n+1}) = \big[(1-\tau_n)^2 f(x_n)^2 + 2(1-\tau_n)f(x_n)\theta_n + \theta_n^2\big] + O\big(f(x_n)^4\big). \tag{1.7}$$

The main part of (1.7) is a quadratic function with respect to θ_n, and we can find its minimum point

$$\theta_n^* = -(1-\tau_n)f(x_n) \tag{1.8}$$

and the minimum value equal to zero. Hence, we have $f_{n+1}^2(\theta_n^*) = O\big(f^4(x_n)\big)$.

From (1.6) and (1.8) we obtain the following equation for τ_n:

$$A_n\tau_n^2 + (1-\tau_n)f(x_n) = 0. \tag{1.9}$$

The root of equation (1.9) that belongs to the interval I_n is given by

$$\tau_n = \frac{1}{2A_n}\left(f(x_n) - f(x_n)\sqrt{1 - \frac{4A_n}{f(x_n)}}\right). \tag{1.10}$$

Since the quantity $\dfrac{4A_n}{f(x_n)}$ is small, one we can use the expansion

$$\sqrt{1 - \frac{4A_n}{f(x_n)}} = 1 - \frac{2A_n}{f(x_n)} - \frac{2A_n^2}{f(x_n)^2} - \frac{4A_n^3}{f(x_n)^3} + O\big(f^4(x_n)\big)$$

in (1.10) and, as a consequence, we obtain $\tau_n = 1 + \dfrac{A_n}{f(x_n)} + \dfrac{2A_n^2}{f(x_n)^2} + O\big(f^3(x_n)\big)$. The values

$$\tau_n = 1 + \frac{A_n}{f(x_n)} \quad \text{and} \quad \tau_n = \left(1 - \frac{A_n}{f(x_n)}\right)^{-1} \tag{1.11}$$

are called the almost optimal choices within the accuracy of $O\big(f^3(x_n)\big)$. In a similar way, the value given by

$$\tau_n = 1 + \frac{A_n}{f(x_n)} + \frac{2A_n^2}{f^2(x_n)} \tag{1.12}$$

is called the almost optimal choice within the accuracy of $O\big(f^4(x_n)\big)$.

The well-known Chebyshev method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} - \dfrac{f''(x_n)f^2(x_n)}{2\big(f'(x_n)\big)^3}$ can be considered as CANM (1.2) with the almost optimal choice (1.12). Similarly, it is easy to show that, in particular, the well-known cubically convergent iterative methods and the fourth-order convergent iterative methods given in [15] lead to iteration (1.2) with the almost optimal parameters given by (1.11) and (1.12), respectively.

1.2. Sufficient conditions for convergence of some well-known iterations. Now we study the convergence of some iterative methods with a given parameter τ_n. To this end, it suffices to obtain some conditions under which the chosen parameter of the method being considered belongs to the τ-region (τ_{reg}) of convergence.

Theorem 2. *We assume that the conditions* (i)-(iv) *in Theorem* 1 *are satisfied. Then, the following iterative methods are convergent if the corresponding numbers* a_n *in* (iv) *obey the condition given in the last column of the table.*

N	Methods	τ_n	$\tau_n \in I_n$
1	Chebyshev method	$1+\dfrac{f''(x_n)}{2f'(x_n)}\dfrac{f(x_n)}{f'(x_n)}$	$a_n < 0.5$
2	Modification of Chebyshev method [15]	$\dfrac{1}{f(x_n)}\left[\left(1+\dfrac{b}{2}\right)f(y_n)+(1+b)f(x_n)-\right.$ $\left.-\dfrac{b}{2}f\left(x_n+\dfrac{f(x_n)}{f'(x_n)}\right)\right]$	$-2 \leqslant b \leqslant 0$, $a_n < 0.5$
3	Method of Weerakoon and Fernando [7]	$\dfrac{2f'(x_n)}{f'(x_n)+f'(y_n)}$	$a_n < 0.5$
4	Method of Frontini and Sormani [7] and Homeier	$\dfrac{f'(x_n)}{f'\left(x_n-\dfrac{f(x_n)}{2f'(x_n)}\right)}$	$a_n < 0.5$
5	Halley-type iteration [9]	$\dfrac{\theta^2 f(x_n)}{(\theta^2-\theta+1)f(x_n)-f\left(x_n-\dfrac{\theta f(x_n)}{f'(x_n)}\right)}$	$a_n < 0.5$
6	Method of Ostrawski and Traub [16]	$\dfrac{f(y_n)-f(x_n)}{2f(y_n)-f(x_n)}$	$a_n < a^*$
7	Method given in [15]	$1+\dfrac{f'(x_n)}{f(x_n)}\dfrac{f(y_n)}{f'(y_n)}$	$a_n < a^*$
8	Method given in [18]	$1+\dfrac{f(y_n)+f(z_n)}{f(x_n)}$	$a_n < 0.5$

Here $y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $z_n = y_n - \dfrac{f(y_n)}{f'(x_n)}$, $0 < \theta < 1$, and $a^* \in (0, 0.5)$ is a root of the cubic polynomial equation $\psi(a) \equiv a^3 - 16a^2 + 24a - 8 = 0$. In [16, 19] it was shown that

$$a_n \leqslant M\gamma^{\sigma^n}, \quad \gamma \in (0,1), \tag{1.13}$$

for the method of Ostrawski and Traub ($\sigma = 4$) and for some modifications of Chebyshev method ($\sigma = 3$). Here σ is the convergence order of these methods. It seems also true for all the methods given in the table that

$$|1-\tau_n| \leqslant ca_n. \tag{1.14}$$

From (1.13) and (1.14) it follows that the rate of convergence of τ_n to 1 as $n \to \infty$ is much greater when the convergence order of the method is higher.

2. Relation between the inexact Newton method and CANM. Let us consider the nonlinear system

$$F(x) = 0, \tag{2.1}$$

where $F : R^n \to R^n$ is a continuously differentiable nonlinear mapping. Among all types of methods for solving the nonlinear equations (2.1), Newton's method is one of the most elementary, popular, and important one. One of the advantages of this method is its local quadratic convergence. However, its computational cost is expensive, particularly when the size of the problem is very large, since in each iteration step the Newton equation

$$F(x_k) + F'(x_k)s_k = 0 \tag{2.2}$$

should be solved.

2.1. An inexact Newton method. To reduce the computational cost of Newton's method, Dembo, Eisenstat and Steihaug proposed the inexact Newton method $F'(x_k)s_k = -F(x_k) + r_k$, $x_{k+1} = x_k + s_k$, $k = 0, 1, \dots$, $x_0 \in D$ [20]. The terms $r_k \in R^n$ represent the residuals of the approximate solutions s_k, i.e., the Newton equation (2.2) is solved inexactly and a step s_k is obtained such that

$$\|r_k\| = \big\|F(x_k) + F'(x_k)s_k\big\| \leqslant \eta_k \big\|F(x_k)\big\|, \tag{2.3}$$

where $\eta_k \in [0, 1)$ is the forcing term. In each iteration step of the inexact Newton method, a real number η_k should be chosen first and, then, an inexact Newton step s_k is obtained by solving the Newton equation approximately with an efficient iterative solver for systems of linear equations. The forcing terms play an important role both in reducing the residual of Newton equations and in the accuracy of method. In particular, if $\eta_k = 0$ for all k, then the inexact Newton method will be reduced to Newton's method. The inexact Newton method (IN), like Newton's method, is locally convergent if $\eta_k \in [0, 1)$ for all k [20].

Theorem 3 [20]. *Assume that the IN iterations converge to x^*. Then, the convergence is superlinear if and only if* $\|r_k\| = o\Big(\big\|F(x_k)\big\|\Big)$ as $k \to \infty$.

Theorem 4 [20]. *Assume that $F : R^n \to R^n$ is continuously differentiable and $x^* \in R^n$ such that $F(x^*) = 0$ and $F'(x^*)$ is not singular. If the sequence x_k generated by IN iterations converges to x^*, then*

(1) *x_k converges to x^* superlinearly when $\eta_k \to 0$;*

(2) *x_k converges to x^* quadratically if $\eta_k = O\Big(\big\|F(x_k)\big\|\Big)$ and $F'(x)$ is Lipschitz continuous at x^*.*

In [21] it is shown that the inexact Newton method, the inexact perturbed Newton method, and the quasi-Newton methods are equivalent.

2.2. CANM for Banach spaces. One of the modifications of Newton's method is the well-known CANM, or damped Newton method,

$$F'(x_k)v_k = -F(x_k), \quad x_{k+1} = x_k + \tau_k v_k, \quad k = 0, 1, 2, \dots, \tag{2.4}$$

where τ_k is the iteration parameter. A suitable choice of the parameter allows one to speed up the convergence and to enlarge the convergence domain. If $\tau_k \equiv 1$ for all k, then CANM is reduced to Newton's method. There exists a closed relation between the CANM and IN iterations. Indeed, CANM can be considered as an IN iteration $F'(x_k)s_k + F(x_k) = r_k = (1 - \tau_k)F(x_k)$, i.e., $\|r_k\| = \eta_k \big\|F(x_k)\big\|$ with $\eta_k = |1 - \tau_k|$. We come to the following local and semi-local convergence results.

Theorem 5. *There exits $\varepsilon > 0$ such that, for any initial approximation x_0 with $\|x_0 - x^*\| \leqslant \varepsilon$, the sequence of* CANM *iterations with a parameter $\tau_k \in (0, 2)$ converges to x^*.*

Theorem 6. *We assume that*

(i) $\big\|F''(x)\big\| \leqslant M$, $x \in D_0$ ($F'(x)^{-1}$ *exists for all* $x \in D_0$),

(ii) $\|F'(x_0)^{-1}\| \leqslant \beta$,

(iii) $\big\|F'(x_0)^{-1}F)(x_0)\big\| \leqslant \eta$, $a_0 = M\beta\eta$,

(iv) $M\big\|F'(x_k)^{-1}\big\|\,\big\|F'(x_k)^{-1}F(x_k)\big\| \leqslant a_k < 2$, $k = 0, 1, \dots$,

and $\tau_n \in I_n = \left(0, \dfrac{-1 + \sqrt{1 + 4a_n}}{a_n}\right) \subseteq (0, 2)$. *Then, the sequence $\{x_n\}$ defined by* (2.4) *converges to a solution x^* of equation* (2.1).

Remark. Assume that the sequence x_k generated by CANM converges to x^*. Then

(1) $\{x_k\}$ converges to x^* superlinearly when $\tau_k \to 1$,

(2) $\{x_k\}$ converges to x^* quadratically if $|1 - \tau_k| = O\Big(\big\|F(x_k)\big\|\Big)$ or $|1 - \tau_k| = O\Big(\big\|F(x_{k-1})\big\|\Big)$.

2.3. Some choices of the iteration parameters. It is easy to show that

$$\frac{\big\|F(x_k)\big\| - |1 - \tau_{k-1}|\,\big\|F(x_{k-1})\big\|}{\big\|F(x_{k-1})\big\|} = O\Big(\big\|F(x_{k-1})\big\|\Big) \quad \text{or} \quad \frac{\big\|F(x_k)\big\| - |1 - \tau_{k-1}|\,\big\|F(x_{k-1})\big\|}{\big\|F(x_k)\big\|} = O\Big(\big\|F(x_{k-1})\big\|\Big).$$

By the above remark, then, one can choose τ_k such that

$$|1 - \tau_k| = \frac{\Big|\big\|F(x_k)\big\| - |1 - \tau_{k-1}|\Big|\,\big\|F(x_{k-1})\big\|}{\big\|F_{k-1}\big\|} \quad \text{or} \quad |1 - \tau_k| = \frac{\Big|\big\|F(x_k)\big\| - |1 - \tau_{k-1}|\,\big\|F(x_{k-1})\big\|\Big|}{\big\|F(x_k)\big\|}, \tag{2.5}$$

which allows the quadratic convergence of CANM. Relations (2.5) can be rewritten in terms of the forcing term η_k as

$$\eta_k = |\alpha_k \eta_{k-1} - 1| \quad \text{or} \quad \eta_k = \frac{|\alpha_k \eta_{k-1} - 1|}{\alpha_k}, \tag{2.6}$$

where $\alpha_k = \dfrac{\|F(x_{k-1})\|}{\|F(x_k)\|}$. Suppose that

$$\|F(x_k)\| \leqslant \eta_{k-1} \|F(x_{k-1})\|, \quad 0 \leqslant \eta_{k-1} < 1. \tag{2.7}$$

Then, $\alpha_k > 1$ and $\alpha_k\ \eta_{k-1} \geqslant 1$. The minimum of the possible choices in (2.6) is $\eta_n = \dfrac{\alpha_k \eta_{k-1} - 1}{\alpha_k}$. If inequality (2.7) is not true, then (2.6) gives $\eta_n = 1 - \eta_{k-1}\alpha_k$. Thus, we have

$$\eta_k = \begin{cases} 1 - \eta_{k-1}\alpha_k, & \text{when} \quad \eta_{k-1}\alpha_k < 1, \\ -\dfrac{1 - \eta_{k-1}\alpha_k}{\alpha_k}, & \text{when} \quad \eta_{k-1}\alpha_k \geqslant 1. \end{cases} \tag{2.8}$$

The second choice in (2.8) allows us to decrease η_k, i.e., $0 < \eta_k < \eta_{k-1}$, while the first choice in (2.8) implies that $0 < \eta_k < 1$. In both these cases we have $0 < \eta_k < 1$. According to (2.3), it is possible that (2.7) is true and thereby the second choice in (2.8) allows us to decrease η_k, i.e., $\eta_k \to 0$ as $k \to \infty$. In terms of τ_k, we have the following choice

$$|1 - \tau_k| = \eta_k. \tag{2.9}$$

From this it follows that, if $0 < \eta_k < 1$ and $\eta_k \to 0$ as $k \to \infty$, we have $0 < \tau_k < 2$ and $\tau_k \to 1$ as $k \to \infty$. Thus, the choice of the iteration parameter is given by (2.8), (2.9) in CANM.

Conclusions. It is shown that the suitable choices of the iteration parameter in CANM allows us not only to enlarge the convergence domain but also to speed-up the convergence.

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References

1. *Aslam Noor M., Ahmad F.* Numerical comparison of iterative methods for solving nonlinear equations // Appl. Math. Comput. 2006. **180**. 167–172.
2. *Aslam Noor M., Inayat Noor Kh.* Three step iterative methods for nonlinear equations // Appl. Math. Comput. 2006. **183**. 322–327.
3. *Aslam Noor M., Inayat Noor Kh.* Some iterative schemes for non-linear equations // Appl. Math. Comput. 2006. **183**. 774–779.
4. *Aslam Noor M., Inayat Noor Kh., Mohyud-Din S.T., Shabbir A.* An iterative method with cubic convergence for nonlinear equations // Appl. Math. Comput. 2006. **183**. 1249–1255.
5. *Chen J.* Some new iterative methods with three order convergence // Appl. Math. Comput. 2006. **181**. 1519–1522.
6. *Ezqnerro J.A., Hernandez M.A.* On Halley-type iterations with free second derivative // J. Comput. Appl. Math. 2004. **170**. 455–459.
7. *Homeier H.H.H.* On Newton-type methods with cubic convergence // Appl. Math. Comput. 2005. **176**. 425–432.
8. *Kou J., Li Y., Wang X.* Modification of Newton method with third-order convergence // Appl. Math. Comput. 2006. **181**. 1106–1111.
9. *Kou J., Li Y., Wang X.* Modified Halley's method free from second derivative // Appl. Math. Comput. 2006. **183**. 704–708.
10. *Kanvar M.V., Kukreja V.K., Singh S.* On some third order iterative methods for solving non-linear equations // Appl. Math. Comput. 2005. **171**. 272–280.
11. *Frontini M., Sormani E.* Some variant of Newton's method with third-order convergence // J. Comput. Appl. Math. 2003. **140**. 419–426.
12. *Weerakon S., Fernando T.I.* A variant of Newton's method with accelerated third-order convergence // Appl. Math. Lett. 2000. **13**. 87–93.
13. *Zhanlav T., Puzynin I.V.* The convergence of iteration based on a continuous analogue of Newton's method // Comput. Math. Math. Phys. 1992. **32**. 729–737.

14. *Ermakov V.V., Kalitkin N.N.* Optimal step and regularization of Newton's method // Zh. Vych. Math. Math. Fiz. 1981. **21**. 491–497.
15. *Zhanlav T., Chuluunbaatar O.* High-order convergent iterative methods for solving nonlinear equations // Bulletin of Peoples' Friendship University of Russia. 2009. N 3. 70–78.
16. *Mi X. and Wang X.* A R-order four iteration in Banach space // J. Comput. Anal. Appl. 2005. **7**. 305–318.
17. *Zhanlav T., Mijiddorj R., and Chuluunbaatar O.* The continuous analogue of Newton's method for finding eigenvalues and eigenvectors of the matrices // Tver University Vestnik. 2008. **14**. 27–37 (in Russian).
18. *Zhanlav T.,Chuluunbaatar O., and Ankhbayar L.* On a Newton-type method with fourth- and fifth-order convergence // Tver University Vestnik (to appear).
19. *Hernandez M.A.* Second-derivative-free variant of the Chebyshev method for nonlinear equations // J. Optim. Theor. Appl. 2000. **104**. 501–515.
20. *Dembo R.S., Eisenstat S.C., and Steihang T.* Inexact Newton methods // SIAM J. Numer. Anal. 1982. **19**. 400–408.
21. *Cătinas. E.* The inexact, inexact perturbed, and quasi-Newton methods are equivalent methods // Math. Comp. 2004. **74**. 291–301.

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