Quantum Field Theory of Neutrino Oscillations in Vacuum and Matter

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Outline

Three generations

About 50 years of experimental research yielded in discovery of three types of neutrinos associated to leptons:

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix} \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \quad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}$$

$$1956 \quad 1962 \quad 2000$$

Are neutrinos massive?

- No direct observation of neutrino mass yet. Only limits:

$$m_{\nu_e} < 2.2 \text{ eV}, \; m_{\nu_\mu} < 170 \text{ keV}, \quad m_{\nu_\tau} < 15.5 \text{ MeV}$$

- From cosmology (model dependent)

$$\sum_{i=\nu_e,\nu_\mu,\nu_\tau} m_i < 1 \text{ eV}$$

- If ν have a mass should ν_e, ν_μ, ν_τ be massive?
  - Not necessarely!

Masses in the SM

In the SM all particles initialy are massless. They acqure their masses due to symmetry breaking interacting with Higgs boson.

- Quarks are a good example:
  - Massless quarks

$$\bar{\mathcal{U}} = (\bar{u}, \bar{c}, \bar{t}) \, , \mathcal{D} = \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

- after symmetry breaking a “mass“ term with non-diagonal matrix $\mathcal{M}$ appears in the SM $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2}\bar{\mathcal{U}}\mathcal{M}\mathcal{D} + \frac{g}{2\sqrt{2}} \bar{\mathcal{U}}\gamma_{\mu} \left(1 - \gamma_5\right) \mathcal{D} \, W^{\mu}$$

and diagonal interaction term

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Masses in the SM

$\mathcal{M}$ can be diagonalized introducing two unitary matrices K, N:

$$\bar{\mathcal{U}} = \bar{U} K^{\dagger},\ \mathcal{D} = N D$$

in a way that:

$$\frac{1}{2}\bar{\mathcal{U}}\mathcal{M}\mathcal{D} = \frac{1}{2}\bar{U} K^{\dagger}\mathcal{M} N D \text{ and } K^{\dagger}\mathcal{M} N \text{ is diagonal}$$

→**The fields U, D are physical because they have a definite mass.**

→**The price is that originally diagonal interaction term becomes non-diagonal:**

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The product of two unitary matrices $V = K^{\dagger}N$ is also a unitary matrix known as CKM matrix.

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Masses in the SM

We used to think about physical massive states of quarks grouped in three doublets:

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix}$$

and having CKM matrix in the non-diagonal interaction sector:

$$\begin{aligned} \mathcal{L} = & \frac{g}{2\sqrt{2}} V_{ud} \bar{u} \gamma_\mu (1 - \gamma_5)\, d\, W^\mu \\ & + \frac{g}{2\sqrt{2}} V_{us} \bar{u} \gamma_\mu (1 - \gamma_5)\, s\, W^\mu, \\ & + \frac{g}{2\sqrt{2}} V_{ub} \bar{u} \gamma_\mu (1 - \gamma_5)\, b\, W^\mu, \ \text{etc} \end{aligned}$$

A note about notations

1. Let me note that we never use a super position like:

$$\psi_{ud} = V_{ud}d + V_{us}s + V_{ub}b$$

to write in short notation:

$$\mathcal{L} = \frac{g}{2\sqrt{2}}\bar{u}\gamma_\mu\left(1-\gamma_5\right)\psi_{ud}$$

2. Let me also note that even if ψ_{ud} was ever used I never heard that one wants to interpret ψ_{ud} as a physical state. Did you ever heard a name like u-th down quark?

Neutrino masses

Neutrino masses appear in the SM in exactly (or to be more precise - VERY similar[1]) way: leading to the very similar mixing matrix V called Pontecorvo-Maki-Nakagawa-Sakata matrix and physical massive neutrinos ν_1, ν_2, ν_3.

Psychological adaptation

However 50 years of research in neutrino field make VERY difficult to think about so called flavour states like: $\nu_e = V_{e1}\nu_1 + V_{e2}\nu_2 + V_{e3}\nu_3$ as about non-physical mixture like $V_{ud}d + V_{us}s + V_{ub}b$

We still like to call both ν_1, ν_2, ν_3 and ν_e, ν_μ, ν_τ as physical sets while in fact ν_1, ν_2, ν_3 are physical and interaction is not diagonal exactly as with quarks.

[1] Have a look for lectures of Andrea Romanino for deeper understanding at http://astronu.jinr.ru/wiki/upload/4/49/Romanino_Lectures_Dubna_2007.pd

Namings

1. We do not use $\psi_{ud} = V_{ud}d + V_{us}s + V_{ub}b$

$$\begin{pmatrix} u \\ d \end{pmatrix} \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \begin{pmatrix} t \\ b \end{pmatrix} \quad \leftarrow \text{massive states}$$

2. However we use $\nu_e = V_{e1}\nu_1 + V_{e2}\nu_2 + V_{e3}\nu_3$

$$\begin{pmatrix} e \\ \nu_1 \end{pmatrix} \quad \begin{pmatrix} \mu \\ \nu_2 \end{pmatrix} \quad \begin{pmatrix} \tau \\ \nu_3 \end{pmatrix} \quad \leftarrow \text{massive states}$$

Assume neutrino have a mass

If neutrinos have a mass then what?

$\longrightarrow$ an interesting effect - neutrino oscillations appears

$\longrightarrow$ We consider this in Quantum Mechanics and QFT

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→ an interesting effect - neutrino oscillations appears
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Naive picture

1. An evolution of a relativistic particle with 4-momenta $p_i = (\sqrt{\mathbf{p}^2 + m_i^2}, \mathbf{p})$ is given by $e^{-ip_i x_i}$ with 4-point $x_i = (t_i^0, \mathbf{x})$
2. If in a reaction was produced $\mid \nu_\alpha \rangle = \sum_i V_{\alpha i} \mid \nu_i(0) \rangle$ then at a position $\mathbf{x}$ the state $\mid \nu_\alpha \rangle$ will be evolved as:

$$\mid \nu_\alpha \rangle = \sum_i V_{\alpha i} e^{-ip_i x_i} \mid \nu_i(0) \rangle$$

3. this may look like a state $\langle \nu_\beta \mid$ with a probability:

$$P_{\alpha\beta} \equiv \mid \langle \nu_\beta(x) \mid \nu_\alpha(0) \rangle \mid^2 = \sum_{ij} V_{\alpha i} V_{i\beta}^* V_{\alpha j}^* V_{j\beta} e^{-i(p_i x_i - p_j x_j)}$$

Periodical transformations = oscillations

In QM we say that ν_α transforms into ν_β after some time at a certain distance $|\mathbf{x}|$ with a periodicity defined by the phase

$$\phi_{ij} = p_i x_i - p_j x_j$$

In the standard approximation $t_i = t_j = |\mathbf{x}|\,, \mathbf{p}_i = \mathbf{p}_j$:

$$\phi_{ij} = (E_i - E_i)\,|\mathbf{x}| - (\mathbf{p}_i - \mathbf{p}_j)\mathbf{x} = \left(\frac{E_i^2 - E_j^2}{E_i + E_j}\right)|\mathbf{x}| \approx \frac{m_i^2 - m_j^2}{2E}\,|\mathbf{x}| \tag{1}$$

Thus ϕ_{ij} depends on E, $|\mathbf{x}|$ and ϕ_{ij} made a ground for the new industry in neutrino physics. How solid is the background?

A small improvement

Let us make a small improvement: take into account that neutrinos with different masses pass the distance $|\mathbf{x}|$ with different times:

$$t_i = \frac{L}{v_i} = |\mathbf{x}| \frac{E_i}{p} \neq t_j = \frac{|\mathbf{x}|}{v_j} = |\mathbf{x}| \frac{E_j}{p}$$

Now:

$$\begin{aligned}\phi_{ij} &= (E_i t_i - E_j t_j) - (\mathbf{p} - \mathbf{p})\mathbf{x} = (\frac{E_i}{v_i} - \frac{E_j}{v_j})\,|\mathbf{x}| \\ &= (\frac{E_i^2}{p} - \frac{E_j^2}{p})\ |\mathbf{x}| = (\frac{p_i^2 + m_i^2}{p} - \frac{p_j^2 + m_j^2}{p})\ |\mathbf{x}| \qquad (2) \\ &= \frac{m_i^2 - m_j^2}{p}\ |\mathbf{x}|\end{aligned}$$

This phase is two times larger!

Naive questions/problems

1. Is the QM description adequate[2] if a small improvement like $v \neq 1$ makes the phase two times larger?
   → which velocity is right?
2. In a decay energy and momentum is conserved but $\nu_\alpha \to \nu_\beta$ means that along the oscillation path neutrino becomes lighter or heavier:
   → is energy-momentum conserved in oscillations?
3. if neutrino momentum is well defined then position is undefined $\delta x \sim \hbar/p = \infty$ then how one can observe oscillations as function of $|\mathbf{x}|$

Questions needs Answers

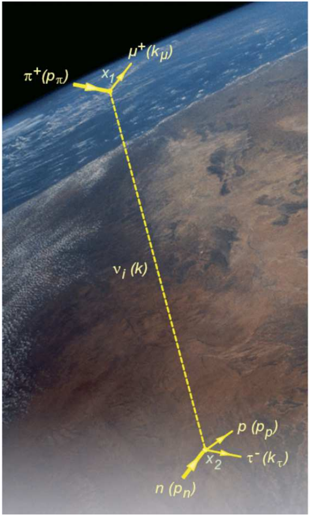
We look for answers to these naive questions/problems going beyond this theory from QM to Quantum Field Theory

[2] QM can be adequte with wave packets too, see Shirokov& V.A.Naumov paper and others

Interaction of neutrino with matter electrons shifts neutrino masses because ν_e interacts with matter due to W and Z exchange, while ν_μ and ν_τ only via Z boson.
Qualitative items:

1. The effect depends on $G_F n_e$
2. Neutral currents are not important as they give same shift to all ν_i
3. The mixing matrix V is redefined in the matter $V \to U_{mat}$

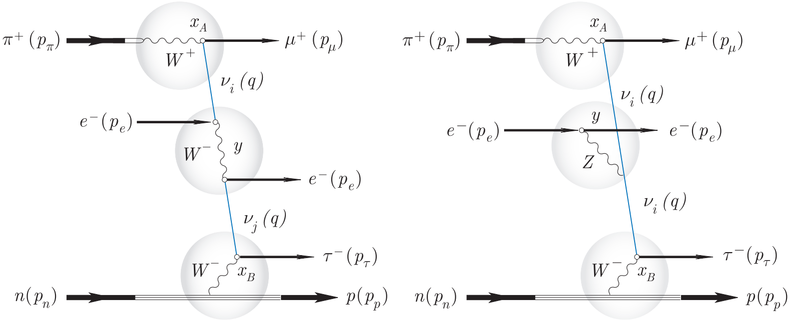
What do we want to compute in QFT?



1. We want to compute in QFT a one-amplitude process with **macroscopically** different production and detection points
2. Neutrino which propagates from x_s to y_d is **virtual**
3. x_s to y_d are localized in space and time
4. between production and detection points there is a **moving matter**

What do we want to compute in QFT?

An example:



Wave packets

Plane wave → Wave packets

Plane wave means no localization in space-time. We need to generalize it to a wave packet.

$$|\mathbf{k}_i, x_i; a_i\rangle \equiv |\mathbf{k}_i, x_i\rangle = \int \frac{d\mathbf{k}\phi_i(\mathbf{k}, \langle\mathbf{k}_i\rangle; x_i)}{(2\pi)^3\sqrt{2k_0}}|\mathbf{k}\rangle \quad (3)$$

$$= \int \frac{d\mathbf{k}a_i(\mathbf{k} - \langle\mathbf{k}_i\rangle)\mathrm{e}^{ik\cdot x_i}}{(2\pi)^3\sqrt{2k_0}}|\mathbf{k}\rangle, \quad (4)$$

where $k = (k_0, \mathbf{k})$, $\quad k_0 = \sqrt{\mathbf{k}^2 + M_i^2}$, with normalization:

$$\int \frac{d\mathbf{k}}{(2\pi)^3} |\phi_i(k, \langle\mathbf{k}_i\rangle)|^2 = \int \frac{d\mathbf{k}}{(2\pi)^3} |a_i(\mathbf{k})|^2 = 1.$$

The initial $|in\rangle$ and final $\langle out|$ states are defined as a direct product of one particle states (3). Therefore the transition amplitude can be written as

$$\mathcal{A} = \langle out|\mathcal{S} - 1|in\rangle \tag{5}$$

exactly as one would write using the plane waves with $|\mathbf{k}_i; x_i\rangle \to |\mathbf{k}\rangle$ substitution. Explicitely the amplitude (5) reads:

$$\mathcal{A} = \int \prod_{i=1}^{N} \frac{d\mathbf{k}_i}{(2\pi)^3[2E_{k_i}]^{1/2}} \phi_i(k_i, \mathbf{k}_i) \prod_{j=1}^{M} \frac{d\mathbf{p}_j}{(2\pi)^3[2E_{p_j}]^{1/2}} \phi_j^*(p_j, \mathbf{p}_j)$$
$$\langle \mathbf{p}_M \ldots \mathbf{p}_1 \, |\mathcal{S} - 1| \, \mathbf{k}_N \ldots \mathbf{k}_1 \rangle, \tag{6}$$

SM inputs

$\mathcal{S}$ matrix

$$\mathcal{S} = \mathrm{T}\left[\mathrm{e}^{i\int[\mathcal{L}_s(x)+\mathcal{L}_m(x)+\mathcal{L}_d(x)]\,dx}\right], \tag{7}$$

Interaction in the source, matter and detector

$$\mathcal{L}_s(x) = -\sqrt{2}G_F J_s^\mu(x)\sum_i V_{\alpha i}\,\bar{\ell}_\alpha(x)\gamma_\mu P_L \nu_i(x) + \text{h.c.}$$

$$\mathcal{L}_d(x) = -\sqrt{2}G_F J_d^\mu(x)\sum_i V_{\beta i}\,\bar{\ell}_\beta(x)\gamma_\mu P_L \nu_i(x) + \text{h.c.},$$

$$\begin{aligned}\mathcal{L}_m(x) = {} & 2\sqrt{2}G_F\sum_{kl} V_{ek}V_{el}^*\,\bar{\nu}_l(x)\gamma^\mu P_L e(x)\cdot\bar{e}(x)\gamma_\mu P_L \nu_k(x) + \\ & + \sqrt{2}G_F\sum_k \bar{\nu}_k(x)\gamma_\mu P_L \nu_k(x)\cdot\bar{e}(x)\gamma^\mu\left(g_L^e P_L + g_R^e P_R\right)e(x) \\ & + \text{h.c.},\end{aligned}$$

Expanding the $\mathcal{S}$ matrix to the second order yields:

$$\begin{aligned}\mathcal{A} =& i^2\langle f| \int dxdy \ \mathrm{T}\left[\mathcal{L}_d(y)\mathrm{e}^{i\int dz\mathcal{L}_m(z)}\mathcal{L}_s(x)\right] |i\rangle \\ =& -\sum_{i,j}\int dxdy \ V^*_{\alpha i}V_{\beta j}\mathrm{e}^{-i(k_s-p_s)x-i(k_d-p_d)y} \\ & \times \overline{\mathcal{M}'}_{dL}\langle \boldsymbol{e}|\mathrm{T}\left[\nu_i(y)\mathrm{e}^{i\int dz\mathcal{L}_m(z)}\bar{\nu}_j(x)\right]|\boldsymbol{e}\rangle\mathcal{M}'_{sR}\end{aligned} \tag{8}$$

with

$$\mathcal{M}'_{sR} = \prod_{i,j}\frac{d\mathbf{k}_i\phi_i(k_i,\langle\mathbf{k}_i\rangle;x_s)}{(2\pi)^3[2E_{k_i}]^{1/2}}\frac{d\mathbf{p}_j\phi^*_i(p_j,\langle\mathbf{p}_j\rangle;x_s)}{(2\pi)^3[2E_{p_j}]^{1/2}}\sqrt{2}G_F P_R J^\mu_s\gamma_\mu\ell_\alpha$$

$$\overline{\mathcal{M}'}_{dL} = \prod_{i,j}\frac{d\mathbf{k}_i\phi_i(k_i,\langle\mathbf{k}_i\rangle;y_d)}{(2\pi)^3[2E_{k_i}]^{1/2}}\frac{d\mathbf{p}_j\phi^*_j(p_j,\langle\mathbf{p}_j\rangle;y_d)}{(2\pi)^3[2E_{p_j}]^{1/2}}\sqrt{2}G_F J^\mu_d\bar{\ell}_\beta\gamma_\mu P_L,$$

where $P_L = 1/2\,(1-\gamma_5)$, $P_R = 1/2\,(1+\gamma_5)$.

In the case of vanishing interaction of neutrino with matter the object

$$G_{ij}(x-y) \equiv \langle \boldsymbol{e}|\mathrm{T}\left[\nu_i(y)\mathrm{e}^{i\int dz\mathcal{L}_m(z)}\bar{\nu}_j(x)\right]|\boldsymbol{e}\rangle$$

is reduced to

$$\langle 0|\nu_i(y)\bar{\nu}_j(x)|0\rangle = \delta_{ij}\int\frac{dq}{(2\pi)^4}\mathrm{e}^{-i(x-y)q}\frac{i}{\hat{q}-m_i}$$

Amplitude

$$\mathcal{A} = -\int dx dy\, \mathrm{e}^{-i(k_s - p_s)x - i(k_d - p_d)y} \overline{\mathcal{M}'}_{dL} \boldsymbol{V}_\beta \boldsymbol{G}(x-y) \boldsymbol{V}^\dagger_\alpha \mathcal{M}'_{sR},$$

where $\boldsymbol{G}(x-y) = ||G_{ij}(x-y)||$, and

$$\boldsymbol{V}^\dagger_\alpha = \begin{pmatrix} V^*_{\alpha 1} \\ V^*_{\alpha 2} \\ V^*_{\alpha 3} \end{pmatrix}, \quad \boldsymbol{V}_\beta = (V_{\beta 1}, V_{\beta 2}, V_{\beta 3}) \tag{9}$$

Virtual neutrino

All non-trivial physics is encoded in neutrino propagator integrated over source and detector space and time.

Details of calculations

Note: No usual QFT δ functions because of localized space-time initial states in the source and the detector

I skip all details of how to take integrals:

✂—✂— ✂— ✂— ✂— ✂— ✂

$$\prod_{i,j}\int\frac{d\mathbf{k}_i\phi_i(k_i,\langle\mathbf{k}_i\rangle;x_s)}{(2\pi)^3[2E_{k_i}]^{1/2}}\int\frac{d\mathbf{p}_j\phi_i^*(p_j,\langle\mathbf{p}_j\rangle;x_s)}{(2\pi)^3[2E_{p_j}]^{1/2}}\int dx\int dy\int dq_0\int d\mathbf{q}$$

in the amplitude $\mathcal{A}$ and integral

$$\int dx_s^0$$

in the $|\mathcal{A}|^2$ **and give you the answer**

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Vacuum case

©C. Y. Cardall, *"Coherence of neutrino flavor mixing in quantum field theory,"* Phys. Rev. D **61**, 073006 (2000)

$$\frac{|\mathcal{A}|^2}{dy_d^0} = d\Gamma(y_d^0) = \int d\mathbf{x}_s \int d\mathbf{y}_d \int \left[\prod_i^{I_S} \frac{d\mathbf{k}_i}{(2\pi)^3}\right] \left[f\left(\mathbf{k}_i, \mathbf{x}_s, x_s^0\right)\right] \times \int \frac{d\mathbf{p}}{(2\pi)^3} f\left(\mathbf{p}, \mathbf{y}_d, y_d^0\right) \, d\Gamma\left(\{\mathbf{k}\}, \{\mathbf{p}\}, \mathbf{x}_s, \mathbf{y}_d\right)$$

- One particle rate $d\Gamma\left(\{\mathbf{k}\}, \{\mathbf{p}\}, \mathbf{x}_s, \mathbf{y}_d\right)$
- distribution functions $f\left(\mathbf{k}_i, \mathbf{x}_s, x_s^0\right)$, $f\left(\mathbf{p}, \mathbf{y}_d, y_d^0\right)$

Decode notations

$$d\Gamma\left(\{\mathbf{k}\},\{\mathbf{p}\},\mathbf{x}_s,\mathbf{y}_d\right) = \int dE_{\mathbf{q}} \left[\frac{d\Gamma\left(\{\mathbf{k}\},E_{\mathbf{q}}\right)}{L^2\, d\Omega_{\mathbf{q}}\, dE_{\mathbf{q}}}\right] \left[P_{\text{mix}}\left(E_{\mathbf{q}},\mathbf{x}_s,\mathbf{y}_d\right)\right] \left[d\sigma\left(\{\mathbf{p}\},E_{\mathbf{q}}\right)\right].$$

$\left[\frac{d\Gamma(\{\mathbf{k}\},E_{\mathbf{q}})}{L^2\, d\Omega_{\mathbf{q}}\, dE_{\mathbf{q}}}\right]$ is the neutrino flux from source to detector at distance L:

$$dE_{\mathbf{q}}\left[\frac{d\Gamma\left(\{\mathbf{k}\},E_{\mathbf{q}}\right)}{L^2\, d\Omega_{\mathbf{q}}\, dE_{\mathbf{q}}}\right] = \frac{1}{L^2}\frac{E_{\mathbf{q}}^2\, dE_{\mathbf{q}}}{(2\pi)^3 2E_{\mathbf{q}}}\left[\prod_i^{I_S}\frac{1}{2E_{\mathbf{k}_i}}\right]\left[\prod_{i'}^{F_S}\int\frac{d\mathbf{k}_{i'}}{(2\pi)^3 2E_{\mathbf{k}_{i'}}}\right] \times \sum_{\text{spins}}\left|\mathcal{M}_s\left(\{\mathbf{k}\},E_{\mathbf{q}}\right)\right|^2 \; (2\pi)^4\delta^4\left(-k_s+q\right)$$

Decode notations

$d\sigma\left(\{\mathbf{p}\}, E_{\mathbf{q}}\right)$ is the cross-section of massless neutrino in the detector:

$$d\sigma(\{\mathbf{p}\}, E_{\mathbf{q}}) = \frac{1}{2E_{\mathbf{q}}2E_{\mathbf{p}}}\left[\prod_{j'}^{F_D}\frac{d\mathbf{p}_{j'}}{(2\pi)^3 2E_{\mathbf{p}_{j'}}}\right]\sum_{\text{spins}}\left|\mathcal{M}_D\left(\{\mathbf{p}\}, E_{\mathbf{q}}\right)\right|^2\ (2\pi)^4\delta^4\left(p_d - q\right)$$

Decode notations

$P_{\text{mix}}\left(E_{\mathbf{q}}, \mathbf{x}_s, \mathbf{y}_d\right)$ - is what one call “oscillation probability“:

$$P_{\text{mix}} = \sum_{i,j} V^*_{\alpha i} V_{\beta i} V_{\alpha j} V^*_{\beta j} \exp\left[-i\frac{(m_i^2 - m_j^2)L}{2E_{\mathbf{q}}} - \frac{(m_i^2 - m_j^2)^2 L^2}{32 E_{\mathbf{q}}^4 \sigma^2}\right]$$

with

- standard phase $\frac{(m_i^2 - m_j^2)L}{2E_{\mathbf{q}}}$ ☑
- new term suppresing oscillations at large L – “coherence term“ $\exp\left[-\frac{(m_i^2 - m_j^2)^2 L^2}{32 E_{\mathbf{q}}^4 \sigma^2}\right]$ with σ - spread in momentum at source and detector ☑

Examining the results

1. At $L \to \infty$ or $\sigma \to 0$ the interference disappears:

$$P_{\text{mix}} = \sum_{i=1}^{3} V_{\alpha i}^* V_{\beta i} \sum_{j=1}^{3} V_{\alpha j} V_{\beta j}^* = \delta_{\alpha\beta}$$

2. Which velocity of neutrino is right:
   $\longrightarrow$ "standard" QM :$v_i = v_j = 1$ or
   $\longrightarrow$ "refined" $v_i = \sqrt{E^2 - m_i^2/E})$?
   - None of them! The most significant contribution comes with an average velocity:

$$\frac{1}{v} = \frac{1}{2}\left(\frac{1}{v_i} + \frac{1}{v_j}\right)$$

How matters the matter?

Neutrino propagator is modified

$$G_{ij}(x-y) \equiv \langle \boldsymbol{e}|\mathrm{T}\left[\nu_i(y)\mathrm{e}^{i\int dz\mathcal{L}_m(z)}\bar{\nu}_j(x)\right]|\boldsymbol{e}\rangle$$
$$\overset{\text{const density}}{=} \int \frac{dq}{(2\pi)^4}\mathrm{e}^{-i(x-y)q} i\boldsymbol{G}(q)$$

$$i\boldsymbol{G}(q) = i\boldsymbol{S}(q) + i\boldsymbol{S}(q)i\boldsymbol{\Omega}i\boldsymbol{S}(q) + i\boldsymbol{S}(q)i\boldsymbol{\Omega}i\boldsymbol{S}(q)i\boldsymbol{S}(q)i\boldsymbol{\Omega}i\boldsymbol{S}(q) + \ldots$$

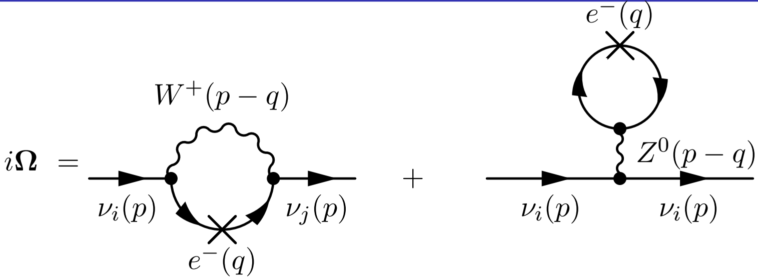
Apparently $\boldsymbol{G}(q)$ satisfies to Dyson-type equation:

$$(\boldsymbol{S}^{-1} - \boldsymbol{\Omega})\boldsymbol{G}(q) = 1, \tag{10}$$

which formal solution is

$$\boldsymbol{G}(q) = (\boldsymbol{S}^{-1} - \boldsymbol{\Omega})^{-1}. \tag{11}$$

Matter effect



$$i\Omega_{ij}^{cc} = -i\sqrt{2}G_F n_e V_{ei} V_{ej}^* \hat{u} P_L,$$

$$i\Omega_{ij}^{nc} = i\delta_{ij}\frac{G_F(g_L^e + g_R^e)}{\sqrt{2}} n_e \hat{u} P_L$$

with

- n_e - density of electrons
- $\hat{u} = \gamma_u(1, \mathbf{u})$ - average four-velocity of matter

Solution for the neutrino propagator

$$P_L \boldsymbol{G} P_R = P_L \left(\boldsymbol{T}_-^\dagger \frac{\hat{q}_-}{\mathcal{P}^2_{\text{m,-}} - \mathbf{q}^2} \boldsymbol{T}_- + \boldsymbol{T}_+^\dagger \frac{\hat{q}_+}{\mathcal{P}^2_{\text{m},+} - \mathbf{q}^2} \boldsymbol{T}_+ \right) P_R$$

with

$$q_\pm = \frac{1}{2}(q_0(1 \pm \mathbf{vn}), q_0(\mathbf{v} \pm \mathbf{n}))$$

- $\mathbf{v}$ - velocity of neutrino
- $\mathbf{n}$ - unit vector of relative velocity between neutrino and matter
- $\mathcal{P}^2_{\text{m},\pm}$ - diagonal “effective“ momentum-squared in the matter (depends on n_e, $q \cdot u$)

Examining the result

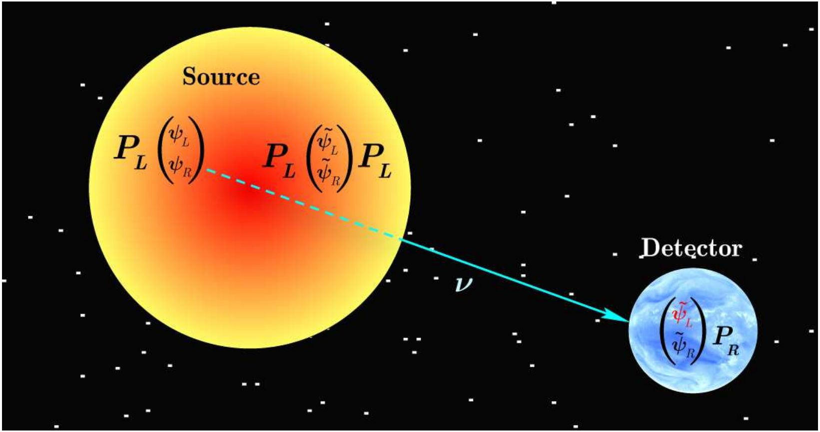
1. Further steps in integrations are exactly like in vacuum case. The only thing which is changed is P_{mix}: matter modifies oscillations
2. In case of matter in rest we get the same answer as standard MSW, while moving matter adds a number of effects:
   - The matter effect increases due to Lorentz boost $n_e \to \gamma_u n_e$
   - The matter effect depends on $1 - \mathbf{uv}$:
     decreases for parallel velocities,
     increases for anti-parallel,
     remains the same for the matter in rest for orthogonal case.
   - new effect: If matter moves faster then neutrino (maybe only academical case, but remember about relic or heavy ν): the oscillations appears between right and left components of neutrino!

Moving Matter

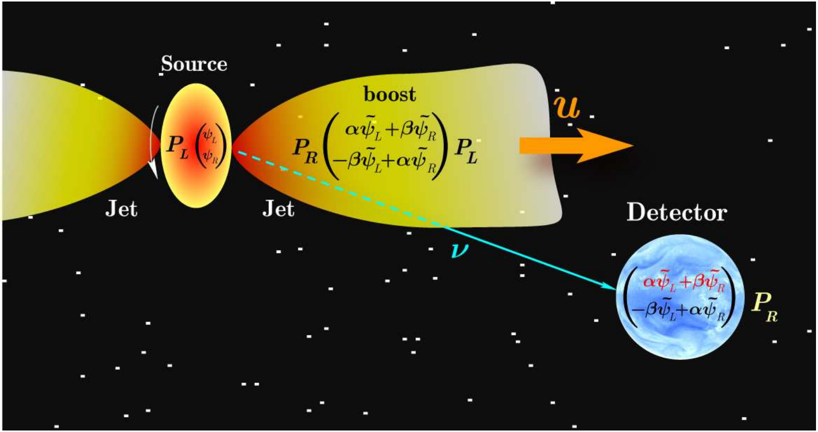
$$P_{\text{mix}} = |\,\alpha_+\,|^2 \sum_{ij} U^{+}_{\beta i} {U^{+}}^{\dagger}_{\alpha i} {U^{+}}_{\beta j} {U^{+}}^{\dagger}_{\alpha j} \mathrm{e}^{i(p_i^+ - p_j^+)L}$$
$$+ |\,\alpha_-\,|^2 \sum_{ij} U^{-}_{\beta i} {U^{-}}^{\dagger}_{\alpha i} {U^{-}}_{\beta j} {U^{-}}^{\dagger}_{\alpha j} \mathrm{e}^{i(p_i^- - p_j^-)L}$$
$$+ \alpha_-\alpha_+ \sum_{ij} U^{+}_{\beta i} {U^{+}}^{\dagger}_{\alpha i} {U^{-}}_{\beta j} {U^{-}}^{\dagger}_{\alpha j} \mathrm{e}^{i(p_i^+ - p_j^-)L}$$
$$+ \alpha_+\alpha_- \sum_{ij} U^{-}_{\beta i} {U^{-}}^{\dagger}_{\alpha i} {U^{+}}_{\beta j} {U^{+}}^{\dagger}_{\alpha j} \mathrm{e}^{i(p_i^- - p_j^+)L}$$

1. Phases $\phi_\pm = (p_i^\pm - p_j^\pm)L$ do not contain neutral currents
2. Phases $\phi_{+-} = (p_i^+ - p_j^-)L$ do contain neutral currents
3. In cases of matter slower than neutrino (most applications) $\alpha_- = 0, \alpha_+ = 1$ and no unusual terms appears

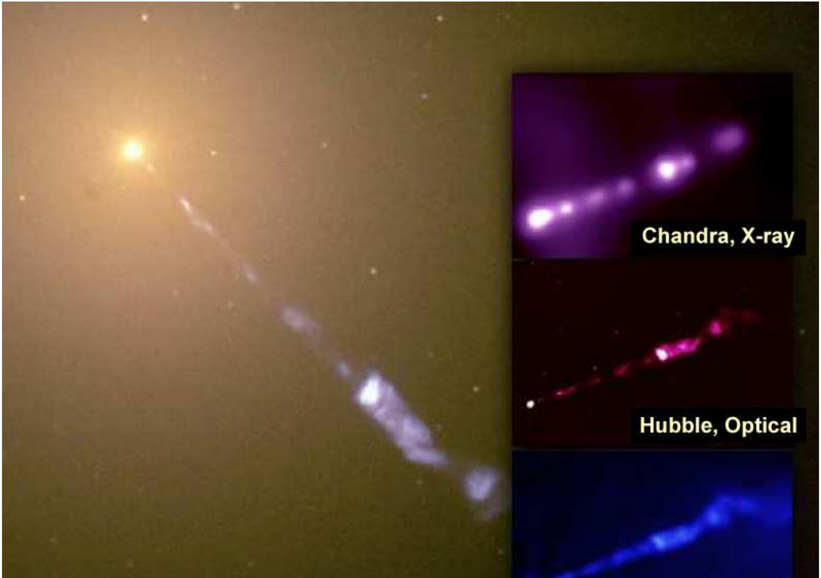
Qualitative Picture. Matter at rest



Qualitative Picture. Fast moving matter



Qualitative Picture. Fast moving matter



Does fast moving matter exists?

http://antwrp.gsfc.nasa.gov/apod/ap000706.html
Explanation: What's causing a huge jet to emanate from the center of galaxy M87? Although the unusual jet was first noticed early in the twentieth century, the exact cause is still debated. The above recently released picture taken by the Hubble Space Telescope shows clear details, however. The most popular hypothesis holds that the jet is created by energetic gas swirling around a massive black hole at the galaxy's center. The result is a 5000 light-year long blowtorch where electrons are ejected outward at near light-speed, emitting eerily blue light during a magnetic spiral. M87 is a giant elliptical galaxy residing only 50 million light-years away in the Virgo Cluster of Galaxies. The faint dots of light surrounding M87's center are large ancient globular clusters of stars.

Note on neutral current

1. For matter in rest neutrino momentum gets a shift due to neutral current like $p \rightarrow p + w_{NC}$ equal for all neutrinos thus no effect on oscillations.
2. for moving matter it is no more true:
    1. Left component gets shift like $p \rightarrow p + (1+ \mid \mathbf{u} - \mathbf{v} \mid) w_{NC}$
    2. Right component gets shift like $p \rightarrow p + (1- \mid \mathbf{u} - \mathbf{v} \mid) w_{NC}$

   These shifts disappear in left-left and right-right terms and do not disappear in left-right and right-left